Use of a map approach for tracking in Scaling FFAs

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Overview

- Typically tracking in FFAs is done using integration of the equations of motion
  - Integrate some form of $\underline{F} = q\, \underline{v} \times \underline{B}$
  - Integrate each time single particles
  - This is fine for tracking few particles through a few turns
  - CPU limited when doing large-scale tracking studies
    - (Synchro)-FFAs have 10s of thousands of turns
    - RK4 uses most of the CPU
- Scaling FFAs are well-suited for taking a transfer map approach for tracking
  - Transfer map is the same at all momenta - it scales!
  - Integrate for one cell; apply for every cell (or turn)
- In this talk
  - Generalised field expansion for horizontal and vertical scaling FFA
  - Determine vector potential and hence Hamiltonian
  - Show that it scales
  - Look at scaling of dynamic aperture (from Runge Kutta for now!)
  - For the future: integrate into Transfer Map

Fields

- Consider a general spiral FFA field - in the midplane

$$
\begin{aligned}
B_r(r, \phi, z=0) &= 0 \\
B_\phi(r, \phi, z=0) &= 0 \\
B_z(r, \phi, z=0) &= f_0(\psi) h(r)
\end{aligned}
$$

Radial scaling

- With

$$
\begin{aligned}
\psi &= \phi - g(r), \\
g &= \tan(\delta) \ln(r/r_0), \\
h(r) &= B_0 \left(\frac{r}{r_0} \right)^k
\end{aligned}
$$

Azimuthal dependence (including spiral angle)

- f_0, B_0 and r_0 are all lattice designer's choice
  - B_0, r_0, determine the reference orbit
  - f_0 is the fringe field length/shape (assume "well behaved" function)

Field expansion

- Apply Maxwell's laws to make an expansion in *z/r*

$$B_z = \sum_{n=0} f_{2n}(\psi)h(r)\left(\frac{z}{r}\right)^{2n}$$

$$B_\phi = \sum_{n=0} f_{2n+1}(\psi)h(r)\left(\frac{z}{r}\right)^{2n+1}$$

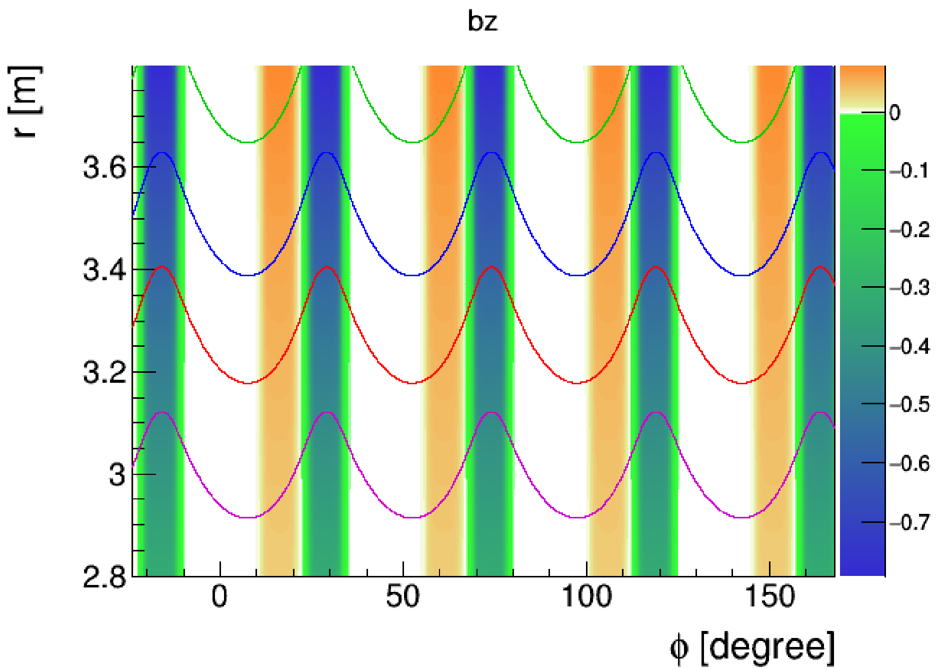
$$B_r = \sum_{n=0}\left[\frac{k-2n}{2n+1}f_{2n} - \tan(\delta)f_{2n+1}\right]h(r)\left(\frac{z}{r}\right)^{2n+1}$$

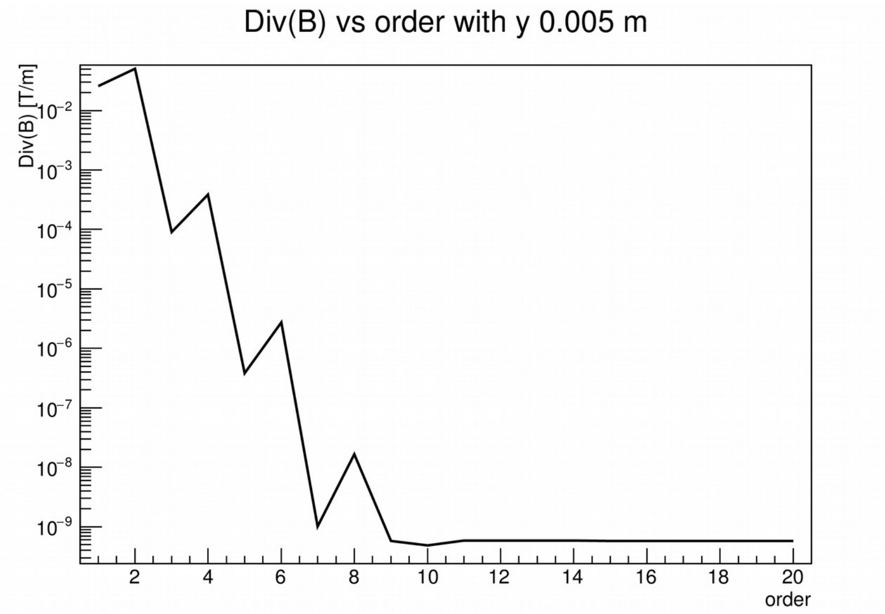
- With

$$f_{2n+1} \quad = \quad \frac{\partial_\psi f_{2n}}{2n+1}.$$

$$f_{2n+2} = \frac{-\frac{(k-2n)^2}{2n+1}f_{2n} + 2(k-2n)\tan(\delta)f_{2n+1} - (1+\tan^2(\delta))\partial_\psi f_{2n+1}}{(2n+2)}.$$

Example field map





- Trajectories scale with momentum
- Increasing order in *z/r* yields better divergence (and curl)

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Vector potential

- Vector potential defined by

$$\vec{B} = \nabla \times \vec{A}.$$

- Choose a gauge with $A_z = 0$

$$A_\phi = \frac{f_0 r h}{k+2} - \sum_n \left[\frac{k-2n}{2n+1} f_{2n} - \tan(\delta) f_{2n+1}\right] h \left(\frac{z}{r}\right)^{2n+2} \frac{r}{2n+2}.$$

$$A_r = -\frac{\tan(\delta) f_0 r h}{k+2} + \sum_n f_{2n+1} h \left(\frac{z}{r}\right)^{2n+2} \frac{r}{2n+2}.$$

Hamiltonian

- Take the usual accelerator Hamiltonian for curved coordinates

$$H = -\left(1+\frac{x}{\rho_0}\right)\left[p_0^2 - (p_x - eA_x)^2 - p_z^2\right]^{1/2} - eA_\phi$$

- Usual equations of motion with ∂_x as partial derivative w.r.t. x

$$\frac{dx}{ds} = \partial_{p_x} H$$
$$\frac{dz}{ds} = \partial_{p_z} H$$
$$\frac{dp_x}{ds} = -\partial_x H$$
$$\frac{dp_z}{ds} = -\partial_z H$$

- Nb: work in coordinate system with constant radius of curvature so that $r = x + \rho_0$

See also Scott Berg FFAG07 “A Hamiltonian Formulation for Spiral-Sector Accelerators”

  - Scott works in spiral coordinate system

Equations of motion

- Consider equations of motion for position

$$\frac{dx}{ds} = \partial_{p_x} H$$
$$\frac{dz}{ds} = \partial_{p_z} H$$

- As usual

$$\frac{dx}{ds} = -(1 + \frac{x}{\rho_0})\frac{p_x}{p_s}$$
$$\frac{dz}{ds} = -(1 + \frac{x}{\rho_0})\frac{p_z}{p_s}$$

Equations of motion

- Consider equations of motion for momentum

$$\frac{dp_x}{ds} = -\partial_x H$$
$$\frac{dp_z}{ds} = -\partial_z H$$

- with

$$\begin{aligned}\partial_x H = &-\frac{1}{\rho_0}\left[p_0^2 - (p_x - eA_x)^2 - p_z^2\right]^{1/2} \\ &+ \left(1 + \frac{x}{\rho_0}\right)\left[p_0^2 - (p_x - eA_x)^2 - p_z^2\right]^{-1/2}(p_x - eA_x)\partial_x A_x \\ &- \partial_x e A_\phi.\end{aligned}$$

$$\begin{aligned}\partial_z H = &\left(1 + \frac{x}{\rho_0}\right)\left[p_0^2 - (p_x - eA_x)^2 - p_z^2\right]^{-1/2}(p_x - eA_x)\partial_z A_x \\ &- \partial_z e A_\phi.\end{aligned}$$

Does it scale?

- Consider scaled coordinates

$$\mathfrak{x} = \alpha x$$
$$\mathfrak{z} = \alpha z$$
$$\rho_{\mathfrak{o}} = \alpha \rho_0$$
$$\Phi = \phi - ln(\alpha) tan(\delta)$$
$$\mathfrak{s} = \Phi \rho_{\mathfrak{o}} = \alpha s - \alpha ln(\alpha) tan(\delta) \rho_0$$
$$\mathfrak{p}_{\mathfrak{x}} = \beta p_x$$
$$\mathfrak{p}_{\mathfrak{z}} = \beta p_z$$
$$\mathfrak{p}_{\mathfrak{s}} = \beta p_s$$

$$\begin{aligned} \psi &= \phi - g(r), \\ g &= \tan(\delta) \ln(r/r_0), \\ h(r) &= B_0 \left(\frac{r}{r_0} \right)^k \end{aligned}$$

- And scaled functions

$$\mathfrak{h}(\mathfrak{x}, \rho_{\mathfrak{o}}) = \alpha^k h(x, \rho_0)$$
$$\mathfrak{g}(\mathfrak{x}, \rho_{\mathfrak{o}}) = \tan(\delta) \ln \left(\alpha \frac{x + \rho_0}{r_0} \right) = \tan(\delta) \left[\ln(\alpha) + \ln \left(\frac{x + \rho_0}{r_0} \right) \right]$$
$$\Psi(\Phi, \mathfrak{x}, \rho_{\mathfrak{o}}) = \Phi - g(\mathfrak{x}, \rho_{\mathfrak{o}}) = \psi(\phi, x, \rho_0)$$
$$\mathfrak{f}_n(\Psi) = f_n(\psi)$$

Scaled functions

- If the equations of motion scale, then after a small step the trajectory will remain in the scaled coordinate system

Phase Space vector

$$\vec{u}(s+ds) = \vec{u}(s) + \frac{d\vec{u}}{ds}ds \quad \text{and} \quad \vec{\mathfrak{u}}(\mathfrak{s}+\mathfrak{d}\mathfrak{s}) = \vec{\mathfrak{u}}(\mathfrak{s}) + \frac{d\vec{\mathfrak{u}}}{d\mathfrak{s}}d\mathfrak{s}$$

Scaled Phase Space vector

- We need the derivative to scale for the machine to be “scaling”
  - Consider e.g. x position

$$\frac{dx}{ds} = -(1+\frac{x}{\rho_0})\frac{p_x}{p_s}$$

$$\mathfrak{x} = \alpha x$$
$$\mathfrak{z} = \alpha z$$
$$\rho_o = \alpha \rho_0$$
$$\Phi = \phi - ln(\alpha)tan(\delta)$$
$$\mathfrak{s} = \Phi\rho_o = \alpha s - \alpha ln(\alpha)tan(\delta)\rho_0$$
$$\mathfrak{p}_\mathfrak{x} = \beta p_x$$
$$\mathfrak{p}_\mathfrak{z} = \beta p_z$$
$$\mathfrak{p}_\mathfrak{s} = \beta p_s$$

Scaled functions

- If the equations of motion scale, then after a small step the trajectory will remain in the scaled coordinate system

$$\vec{u}(s+ds) = \vec{u}(s) + \frac{d\vec{u}}{ds}ds \quad \text{and} \quad \vec{\mathfrak{u}}(\mathfrak{s}+\mathfrak{d}\mathfrak{s}) = \vec{\mathfrak{u}}(\mathfrak{s}) + \frac{d\vec{\mathfrak{u}}}{d\mathfrak{s}}d\mathfrak{s}$$

- By reference to the equations of motion we require

$$\frac{\beta}{\alpha}\partial_x A_\phi(x,z) = \partial_{\mathfrak{x}} A_\phi(\mathfrak{x},\mathfrak{z})$$
$$\frac{\beta}{\alpha}\partial_x A_x(x,z) = \partial_{\mathfrak{x}} A_x(\mathfrak{x},\mathfrak{z})$$
$$\frac{\beta}{\alpha}\partial_z A_\phi(x,z) = \partial_{\mathfrak{z}} A_\phi(\mathfrak{x},\mathfrak{z})$$
$$\frac{\beta}{\alpha}\partial_z A_x(x,z) = \partial_{\mathfrak{z}} A_x(\mathfrak{x},\mathfrak{z})$$
$$\beta A_x(x,z) = A_x(\mathfrak{x},\mathfrak{z})$$

$$\begin{aligned}\partial_x H = &-\frac{1}{\rho_0}\left[p_0^2 - (p_x - eA_x)^2 - p_z^2\right]^{1/2} \\ &+ \left(1+\frac{x}{\rho_0}\right)\left[p_0^2 - (p_x - eA_x)^2 - p_z^2\right]^{-1/2}(p_x - eA_x)\partial_x A_x \\ &- \partial_x eA_\phi. \\ \partial_z H = &\left(1+\frac{x}{\rho_0}\right)\left[p_0^2 - (p_x - eA_x)^2 - p_z^2\right]^{-1/2}(p_x - eA_x)\partial_z A_x \\ &- \partial_z eA_\phi.\end{aligned}$$

Scaled functions

- Try A_x

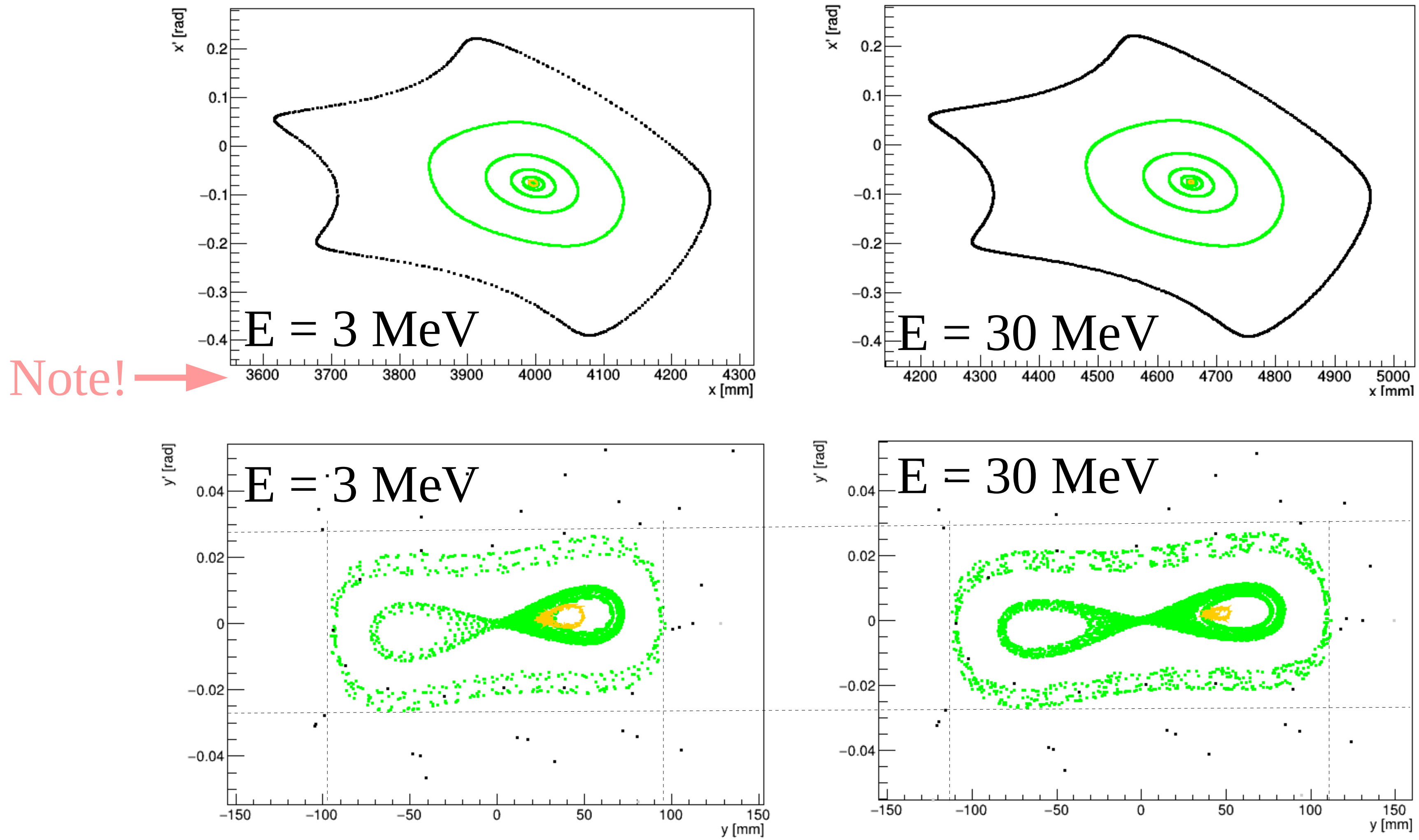
$$\begin{aligned} A_x(\mathfrak{x},\mathfrak{z},\rho_o) &= \sum_n \mathfrak{f}_{2n+1}\mathfrak{h}\left(\frac{\mathfrak{z}}{\mathfrak{x}+\rho_o}\right)^{2n+1}\frac{\mathfrak{z}}{2n+2} \\ &= \sum_n f_{2n+1}h\alpha^k\left(\frac{z}{x+\rho_0}\right)^{2n+1}\alpha\frac{z}{2n+2} \\ &= \alpha^{k+1}A(x,z,\rho_0) \end{aligned}$$

- … and so on for the derivatives, with $\beta = \alpha^{k+1}$

$$\begin{aligned} \partial_{\mathfrak{x}}A_\phi(\mathfrak{x},\mathfrak{z}) &= \alpha^k\partial_x A_\phi(x,z) \\ \partial_{\mathfrak{x}}A_x(\mathfrak{x},\mathfrak{z}) &= \alpha^k\partial_x A_x(x,z) \end{aligned}$$

$$\begin{aligned} \partial_{\mathfrak{z}}A_\phi(\mathfrak{x},\mathfrak{z}) &= \alpha^k\partial_z A_\phi(x,z) \\ \partial_{\mathfrak{z}}A_x(\mathfrak{x},\mathfrak{z}) &= \alpha^k\partial_z A_x(x,z). \end{aligned}$$

Scaled functions



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Horizontal FFA

- Trajectory scales at all orders, perfectly
  - Not just linear
- Trajectory scales in horizontal and vertical direction
  - Geometric acceptance of the transfer map increases with momentum
- Integration is done here using RK4
  - Not *perfectly* identical
    - Stepping in time
    - Do not scale the time step
    - Use same max time for number of steps
  - ToDo: Implement transfer map (use OPAL Lie algebra routines?)

Vertical FFA

- For a vertical FFA we can use similar path
  - Field expansion (see e.g. Machida et al., *Optics design of vertical excursion fixed-field alternating gradient accelerators*, PRAB 24, (2021)

$$B_x = \sum_n B_0 \exp(mz) \frac{1}{m} \partial_x f_n y^n$$

$$B_y = \sum_n B_0 \exp(mz) \frac{n+1}{m} f_{n+1} y^n$$

$$B_z = \sum_n B_0 \exp(mz) f_n y^n$$

$$f_0 = f(x)$$

$$f_1 = 0$$

$$f_{n+2} = \frac{-1}{(n+2)(n+1)} [\partial_x^2 f_n + m^2 f_n].$$

y is horizontal
x is longitudinal
z is vertical

Vector potential

- Vector potential

$$A_x = -B_0 \sum_n \exp(mz) f_n \frac{y^{n+1}}{n+1}$$

$$A_y = 0$$

$$A_z = B_0 \sum_n \frac{\exp(mz)}{m} \partial_x f_n \frac{y^{n+1}}{n+1}$$

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Vertical scaling transformation

- Follow the same route to verify that the machine is scaling, but now

$$\mathfrak{x} = x$$

$$\mathfrak{y} = y$$

$$\mathfrak{z} = z + z_0$$

$$\rho_{\mathfrak{o}} = \rho_0$$

$$\mathfrak{p}_{\mathfrak{x}} = \beta p_x$$

$$\mathfrak{p}_{\mathfrak{y}} = \beta p_y$$

$$\mathfrak{p}_{\mathfrak{z}} = \beta p_z$$

Vertical FFA

- Follow the same route to verify that the machine is scaling, but now we require

$$\beta \partial_y A_x(x,z) = \partial_{\mathfrak{y}} A_x(\mathfrak{x},\mathfrak{z})$$
$$\beta \partial_y A_z(x,z) = \partial_{\mathfrak{y}} A_z(\mathfrak{x},\mathfrak{z})$$
$$\beta \partial_z A_x(x,z) = \partial_{\mathfrak{z}} A_x(\mathfrak{x},\mathfrak{z})$$
$$\beta \partial_z A_z(x,z) = \partial_{\mathfrak{z}} A_x(\mathfrak{x},\mathfrak{z})$$
$$\beta A_z(x,z) = A_z(\mathfrak{x},\mathfrak{z})$$

$$\begin{aligned}\partial_y H = &-\frac{1}{\rho_0}\left[p_0^2-(p_z-eA_z)^2-p_y^2\right]^{1/2} \\ &+\left(1+\frac{y}{\rho_0}\right)\left[p_0^2-(p_z-eA_z)^2-p_y^2\right]^{-1/2}(p_z-eA_z)\partial_y A_z \\ &-\partial_y eA_x. \\ \partial_z H = &\left(1+\frac{y}{\rho_0}\right)\left[p_0^2-(p_z-eA_z)^2-p_z^2\right]^{-1/2}(p_z-eA_z)\partial_z A_z \\ &-\partial_z eA_x.\end{aligned}$$

E.g.

$$\begin{aligned}\partial_{\mathfrak{y}} A_x(\mathfrak{x},\mathfrak{z}) &= -\partial_y B_0 \sum_n \exp\left(m(z+z_0)\right) f_n \frac{y^{n+1}}{n+1} \\ &= -\exp(mz_0)\partial_y A_x(x,z)\end{aligned}$$

$$A_x = -B_0 \sum_n \exp(mz) f_n \frac{y^{n+1}}{n+1}$$
$$A_y = 0$$
$$A_z = B_0 \sum_n \frac{\exp(mz)}{m} \partial_x f_n \frac{y^{n+1}}{n+1}$$

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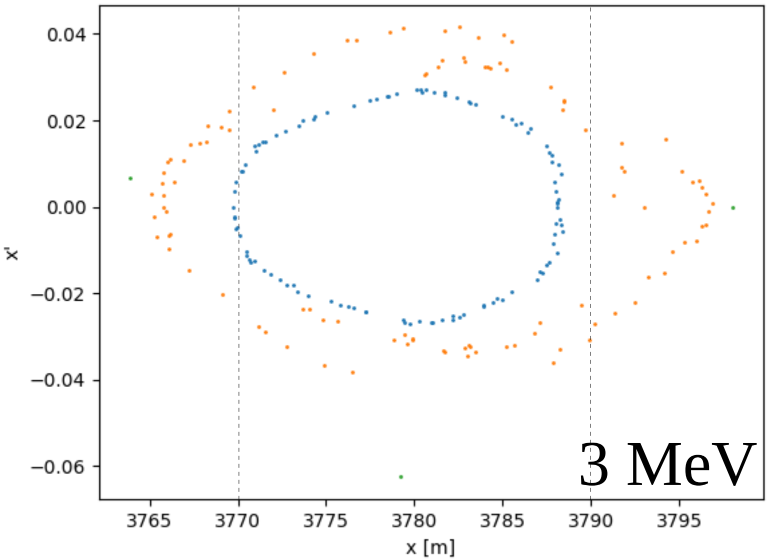
Scaled DA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0.00
−0.02
−0.04
−0.06
x'
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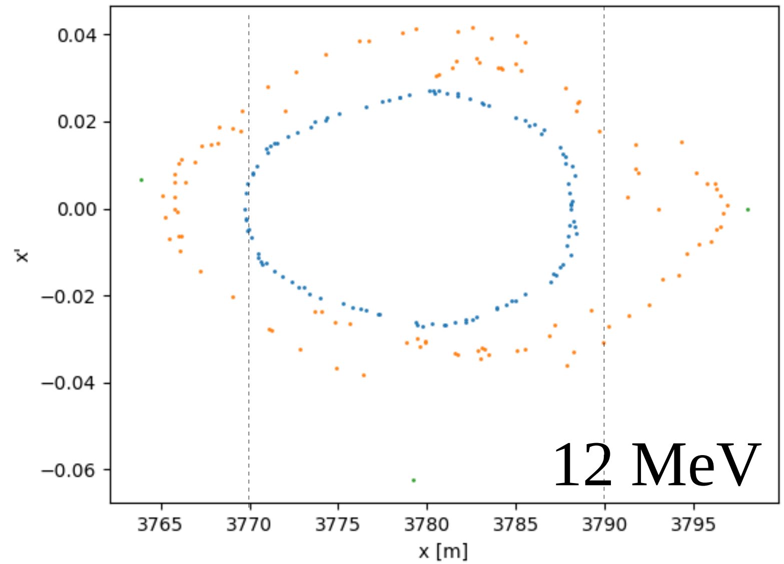
3770
3775
3780
3785
3790
3795
x [m]
3 MeV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0.02
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x'
3765
3770
3775
3780
3785
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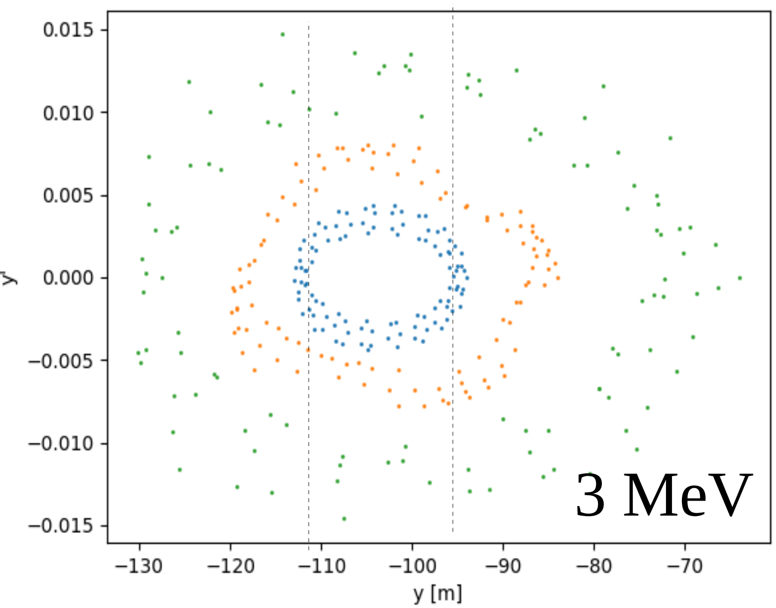
3795
x [m]
12 MeV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0.005
0.000
−0.005
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−0.015
y'
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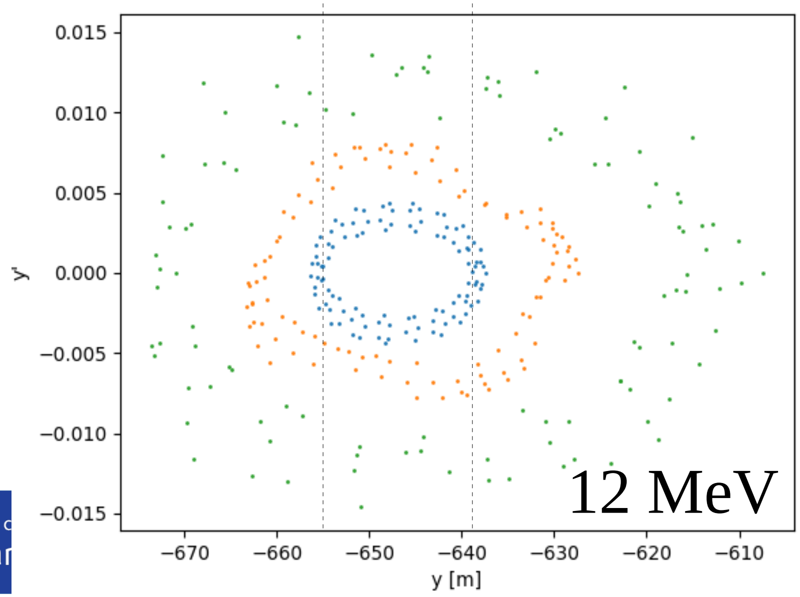
−120
−110
−100
−90
−80
−70
y [m]
3 MeV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0.010
0.005
0.000
−0.005
−0.010
−0.015
y'
−670
−660
−650
−640
−630
−620
−610
y [m]
12 MeV

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- Trajectory scales at all orders, perfectly
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- Integration is done here using RK4
  - ToDo: Implement transfer map (use OPAL Lie algebra routines?)