Hagedorn with a twist in $\mathcal{N} = 4$ SYM

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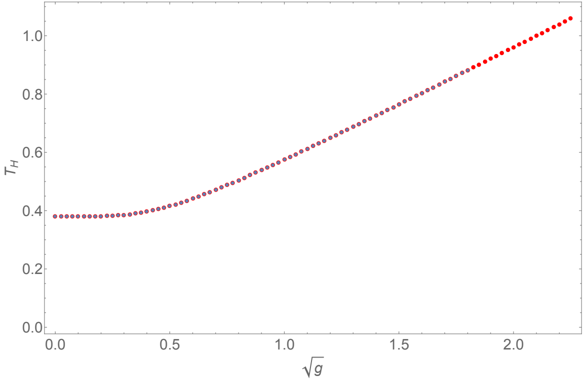
Introduction

- The quantum spectral curve (QSC) is a powerful tool for computing physical observables in planar $\mathcal{N}=4$ SYM for all 't Hooft couplings.
- Recently, the QSC has been used to study its Hagedorn temperature.

Harmark and Wilhelm '17, '18, '21 •; Ekhammar, JAM, Thull '23, '23 •



$g = \frac{\sqrt{\lambda}}{4\pi}$

$$T_H \approx 0.39894 g^{1/2} + 0.15916 - 0.00865 g^{-1/2} + 0.0356 g^{-1} + \mathrm{O}(g^{-3/2})$$

Introduction

We know the first few analytic coefficients

$$T_H = \frac{1}{\sqrt{2\pi}} g^{1/2} + \frac{1}{2\pi} - \frac{5-8\ln 2}{8\sqrt{2\pi^3}} g^{-1/2} + \frac{45}{128\pi^2} g^{-1} + \mathrm{O}(g^{-3/2})$$

- $\frac{1}{\sqrt{2\pi}}$ – Result from flat space limit in AdS/CFT
- $\frac{1}{2\pi}$ – Curvature correction from a winding string on a compactified Euclidean time direction Maldacena '21; Urbach '22
- The higher terms come from next order curvature corrections and a shift in the zero-point string world-sheet. Ekhammar, JAM, Thull '23, '23; Harmark '24
- In this talk I discuss generalizations to include chemical potentials?
- Why do we care?
  - Explore more of phase space
  - Interesting behavior in certain limits where we find simplifications at strong coupling
  - More data to work toward a full world-sheet solution

Hagedorn at $\lambda = 0$

- Single trace local operator: $\mathcal{O}_L(0) = \mathrm{Tr}[\Phi_1\Phi_2\ldots\Phi_L(0)]$
- Maps to states on $R \times S^3$ (Radius set to 1)

$$\mathcal{Z}_1(T) = \sum_{L=2}^{\infty} \sum_{\mathcal{O}_\mathrm{L}} e^{-\Delta/T}$$

- Sundborg 1999: Φ_i: "beads on a necklace". $\Phi_i = \phi^I,\ \psi^a,\ \bar{\psi}_a,\ F_{\mu\nu}$ and their symmetrized covariant derivatives.
- $$Z_{bead}(y) = Z_\phi(y) + Z_\psi(y) + Z_F(y) \qquad y \equiv e^{-1/(2T)}$$
$$= \frac{6(y^2 - y^6)}{(1-y^2)^4} + \frac{16y^3}{(1-y^2)^3} + \frac{2y^4(3-y^2)}{(1-y^2)^3} = \frac{2y^2(3-y)}{(1-y)^3}$$
$$\mathcal{Z}_1 \sim \sum_{L=1}^{\infty} \frac{1}{L} \left(Z_{bead}(y)\right)^L = -\log(1 - Z_{bead}(y))$$
- Diverges for $Z_{bead}(y) = 1 \Rightarrow y = 2 - \sqrt{3} \Rightarrow T_H = \frac{1}{2\log(2+\sqrt{3})}$
- Note that T_H is set by operators with $L \to \infty$

R-charge chemical potentials

- Turn on a chemical potentials for the R-charges Yamada and Yaffe '06

$$\mathcal{Z} = \sum_{\text{states}} e^{-(\Delta - \mu_1 Q_1 - \mu_2 Q_2 - \mu_3 Q_3)/T},$$

$$\mu_1 = \mu; \mu_2 = \mu_3 = 0: \qquad \mathcal{Z}_1 = \sum_{L=2}^{\infty} e^{-(\Delta - \mu Q)/T}$$

$$Z, \bar{Z}:\ Q = \pm 1; \quad X, Y:\ Q = 0; \quad \psi^a, \bar{\psi}_a:\ Q = \pm 1/2$$

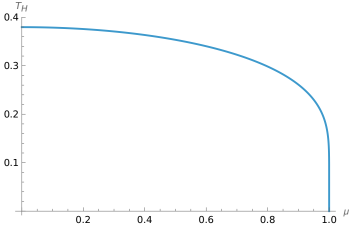
- $\lambda = 0$: $y = e^{-1/(2T)}$ and $z = e^{\mu/2T}$:

$$Z_{bead}(y,z) = \frac{(4 + z^2 + z^{-2})(y^2 + y^4) + 8(z + z^{-1})y^3 + 6y^4 - 2y^6}{(1 - y^2)^3}$$

$Z_{bead}(y,z) = 1 \ \Rightarrow$

$\cosh \frac{1}{2T_H}(\cosh \frac{1}{2T_H} - \cosh \frac{\mu}{2T_H}) = 2$

$\mu \to 1$: $T_H \approx \left(\log \frac{8}{1-\mu}\right)^{-1}$



Spin chemical potentials

- Turn on a chemical potentials for the spins

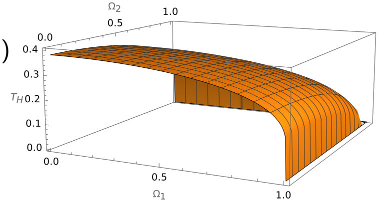
$$\mathcal{Z}_1 = \sum_{L=2}^{\infty} e^{-(\Delta - \Omega_1 S_1 - \Omega_2 S_2)/T}$$

$$Z_{bead} = \frac{6y^2(1-y^4)+4y^3(1-y^2)(\chi_{\frac{1}{2}}(z)+\chi_{\frac{1}{2}}(w))+y^4(\chi_1(z)+\chi_1(w))-2y^6(\chi_{\frac{1}{2}}(z)\chi_{\frac{1}{2}}(w))+2y^8}{(1-y^2 zw)(1-y^2\frac{z}{w})(1-y^2\frac{w}{z})(1-y^2\frac{1}{zw})}$$

$$z = e^{\frac{\Omega_1+\Omega_2}{2T}}\,, \quad w = e^{\frac{\Omega_1-\Omega_2}{2T}}\,, \quad \chi_{\frac{1}{2}}(z) = z + z^{-1}\,, \quad \chi_1(z) = z^2 + 1 + z^{-2}$$

$Z_{bead}(y,z) = 1 \ \Rightarrow$

$$\cosh\frac{1}{2T_H}\left(\cosh\frac{1}{T_H} - \cosh\frac{\Omega_1}{T_H} - \cosh\frac{\Omega_2}{T_H} - 3\right) = 4\cosh\frac{\Omega_1}{2T_H}\cosh\frac{\Omega_2}{2T_H}$$



$\Omega_1 \to 1$: $T_H \sim (1-\Omega_2)\left(\log\frac{1}{1-\Omega_1}\right)^{-1}$

$\Omega_1 = \Omega_2 \equiv \Omega \to 1$: $T_H \sim \frac{1-\Omega}{\log 2}$

R-charge chemical potentials at strong coupling: $\lambda \gg 1$

- $ds^2 = (1+R^2)d\tau^2 + \frac{dR^2}{1+R^2} + R^2 d\Omega_3^2 + (1-Z^2)(A_\mu dx^\mu + d\psi)^2 + \frac{dZ^2}{1-Z^2} + Z^2 d\Omega_3^2$

  $\tau \equiv \tau + \beta\,, \qquad A_\mu dx^\mu = i\,\mu\, d\tau$

- String winding mode around the thermal circle: χ

$$\int d^{10}X \sqrt{g}\left(\nabla^\mu \chi \nabla_\mu \chi + m^2(R,Z)\chi^2\right)\,,$$

$$m^2(R,Z) = (1+R^2)\left(\frac{\beta}{2\pi\alpha'}\right)^2 - (1-Z^2)\left(\frac{\beta\mu}{2\pi\alpha'}\right)^2 + C\,.$$

$C = -\frac{2}{\alpha'} + \Delta C, \qquad \Delta C = \frac{\log 2}{2\pi^2\alpha'} R_{\mu\nu} V^\mu V^\nu + \mathcal{O}(\alpha')$ Harmark '24

$$V^0 = \beta\,, \quad R_{00} = -4(1+\mu^2) - 4(R^2 - Z^2)$$

$\Delta C = \frac{\beta^2}{2\pi^2\alpha'} 4\log(2)[(1+\mu^2) + (R^2 - \mu^2 Z^2)] + \mathcal{O}(\alpha')$

- Tune χ to be massless $\Longrightarrow \chi(R,Z)$

$$\mu \lesssim 1: \quad T_H = \frac{\sqrt{(1-\mu^2)}}{\sqrt{2\pi}} g^{1/2} + \frac{1+\mu}{2\pi} + \frac{(1+\mu)^2 - (1+\mu^2)4\log(2)}{4\sqrt{2\pi^3(1-\mu^2)}} g^{-1/2}$$

$$\frac{1}{\alpha'} = \sqrt{\lambda} \qquad\qquad - \frac{3(1-\mu^2)(1+\mu)}{256\pi^2\mu} g^{-1} + \mathrm{O}(g^{-3/2})$$

$$\mu \to 1: \; \tilde{g} = \frac{1}{4} g(1-\mu^2), \quad \frac{T_H}{2} = \frac{1}{\sqrt{2\pi}} \tilde{g}^{1/2} + \frac{1}{2\pi} + \frac{1 - 2\log(2)}{4\sqrt{2\pi^3}} \tilde{g}^{-1/2} + \mathrm{O}(\tilde{g}^{-3/2})$$

Spin chemical potentials at strong coupling: $\lambda \gg 1$

$$ds^2 = (1+x^2+y^2)d\tau^2 + dx^2 + dy^2 - \frac{(xdx+ydy)^2}{1+x^2+y^2}$$
$$+ x^2(A^{(1)}_\mu dx^\mu + d\phi_1)^2 + y^2(A^{(2)}_\mu dx^\mu + d\phi_2)^2$$

$$A^{(1)}_\mu dx^\mu = i\,\Omega_1 d\tau \qquad A^{(2)}_\mu dx^\mu = i\,\Omega_2 d\tau$$

- String winding mode : $\chi(x,y)$

$$m^2(x,y) = \left(\frac{\beta}{2\pi\alpha'}\right)^2 \left(1 + \left(1-\Omega_1^2\right)x^2 + \left(1-\Omega_2^2\right)y^2\right) + C$$

$$C = -\tfrac{2}{\alpha'} - \tfrac{\beta^2}{2\pi^2\alpha'} 4\log(2)[1+(1-\Omega_1^2)x^2 + (1-\Omega_2^2)y^2] + \mathcal{O}(\alpha')$$

$$T_H = \frac{1}{\sqrt{2\pi}} g^{\frac{1}{2}} + \frac{\sqrt{1-\Omega_1^2}+\sqrt{1-\Omega_2^2}}{4\pi} + \frac{6+\left(\sqrt{1-\Omega_1^2}+\sqrt{1-\Omega_2^2}\right)^2 - 16\log(2)}{16\sqrt{2\pi^3}} g^{-\frac{1}{2}}$$
$$+ \frac{1}{128\pi^2}\left(\frac{23-24\Omega_1^2}{\sqrt{1-\Omega_1^2}} + \frac{23-24\Omega_2^2}{\sqrt{1-\Omega_2^2}} - \frac{2}{\sqrt{1-\Omega_1^2}+\sqrt{1-\Omega_2^2}}\right) g^{-1} + \mathrm{O}(g^{-\frac{3}{2}})$$

Smoothly connects to $\Omega_{1,2} = 0$.

The two leading terms with $\Omega_2 = 0$ match Seitz & Urbach '25

Spin chemical potentials at strong coupling: $\lambda \gg 1$

Needs to be modified as $\Omega_1 \to 1$ and/or $\Omega_2 \to 1$

$\Omega_1 \to 1$: $m^2(x,y) =$

$$m^2(y) = \left(\frac{\beta}{2\pi\alpha'}\right)^2 \left(1 + (1-\Omega_2^2)y^2\right) - \frac{2}{\alpha'} - \frac{\beta^2}{2\pi^2\alpha'} 4\log(2)\left(1 + (1-\Omega_2^2\right) y^2) + \mathcal{O}(\alpha')$$

$$T_H = \frac{1}{\sqrt{2\pi}} g^{\frac{1}{2}} + \frac{\sqrt{1-\Omega_2^2}}{4\pi} + \frac{5-\Omega_2^2-16\log(2)}{16\sqrt{2\pi^3}} g^{-\frac{1}{2}} + \frac{13-16\Omega_2^2}{128\pi^2\sqrt{1-\Omega_2^2}} g^{-1} + \mathrm{O}(g^{-\frac{3}{2}})$$

$$\Omega_{1,2} \to 1 : m^2(x,y) = m^2 = \left(\frac{\beta}{2\pi\alpha'}\right)^2 - \frac{2}{\alpha'} - \frac{\beta^2}{2\pi^2\alpha'} 4\log(2) + \mathcal{O}(\alpha')$$

$$T_H = \frac{1}{\sqrt{2\pi}} g^{1/2} - \frac{\log(2)}{\sqrt{2\pi^3}} g^{-1/2} + \mathrm{O}(g^{-3/2})$$

The twisted quantum spectral curve (QSC)

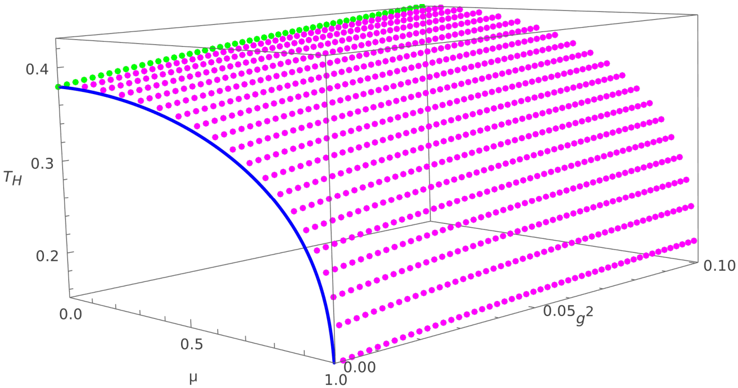
- The QSC is a set of difference equations for "Q-functions" that arise from the Bethe ansatz for $\mathcal{N} = 4$ SYM.

$$\mathbf{P}_a(u),\ \mathbf{Q}_i(u) \text{ and } Q_{a|i}(u), \qquad a, i = 1, 2, 3, 4$$

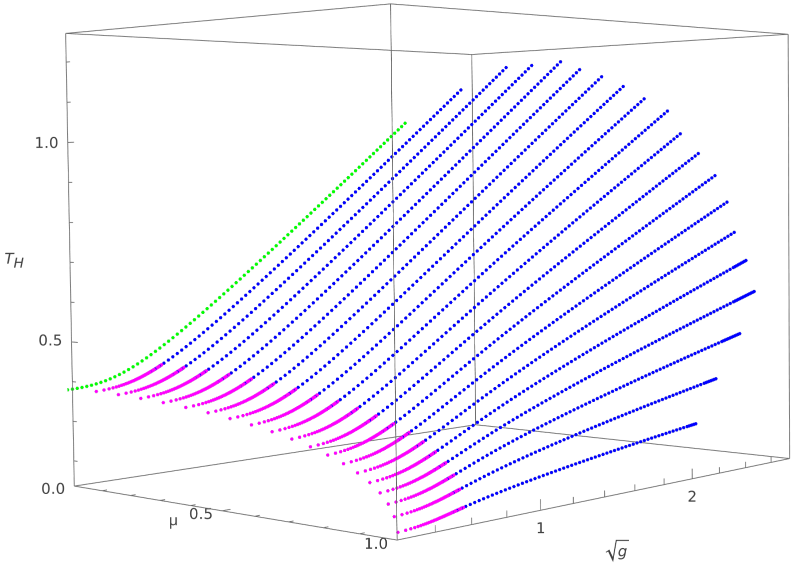
$$Q^+_{a|i} - Q^-_{a|i} = \mathbf{P}_a \mathbf{Q}_i, \qquad \mathbf{Q}_i = -\mathbf{P}^a Q^+_{a|i}, \qquad f(u)^\pm = f(u \pm \frac{i}{2})$$

- $\mathbf{P}_a(u)$ has a single square-root branch-cut on $[-2g, 2g]$
  $\mathbf{Q}_a(u)$ has cuts on $[-2g - i\,n, 2g - i\,n]$, $n \geq 0$
- Asymptotics for large u are determined by setting $T = T_H$
- Chemical potentials add "twists" to the asymptotics
- Impose a "gluing" condition along $[-2g, 2g]$: $\overline{\mathbf{Q}_i(u)} = (-1)^{i+1}\tilde{\mathbf{Q}}_i(u)$
- Can be solved perturbatively for small g^2. Still very computationally intensive
- Only know how to solve numerically for large g

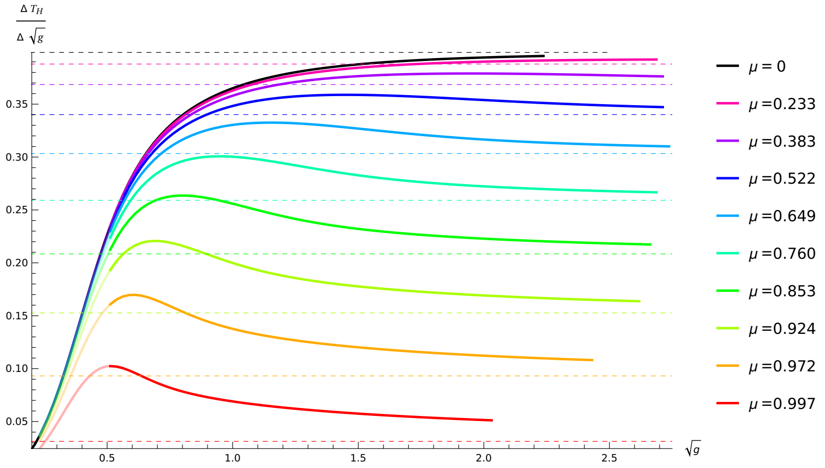
QSC results: Weak coupling (Single R-charge)



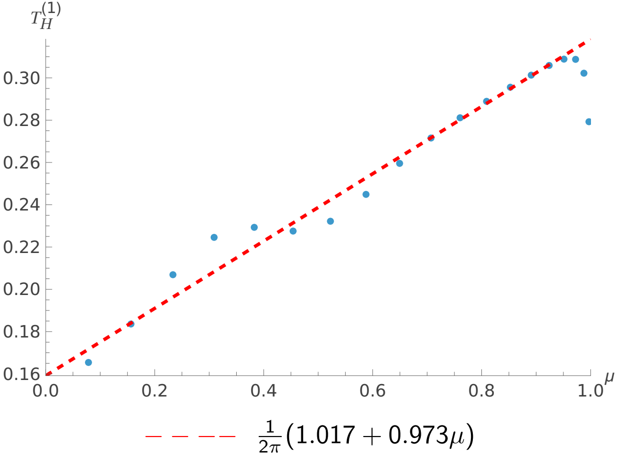
QSC results: Weak to strong coupling



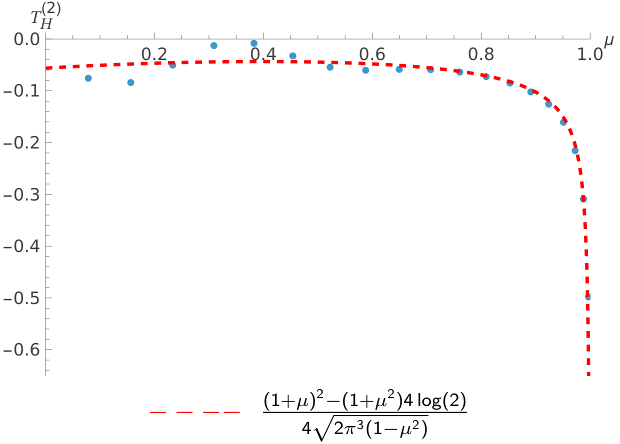
QSC results: $\frac{dT_H}{d\sqrt{g}}$



QSC results: Coefficient of the first subleading term



Coefficient of the second subleading term



Conclusions

- At zero coupling the contribution of chemical potentials to the Hagedorn temperature is computable
- Also true at strong coupling for the first few terms in the $1/g$ expansion using indirect world-sheet arguments.
- Partially verified using the twisted QSC at strong coupling for a single R-charge chemical potential.
- Recent progress in the QSC should greatly improve the numerics and help improve the predictions from the string dual. Ekhammar, Gromov and Ryan '24
- We still do not know how to directly compute these corrections from the string world-sheet because of the usual problems of the background Ramond-Ramond fields

Thanks!