Regge Trajectories from the Adjoint Sector of Matrix Quantum Mechanics

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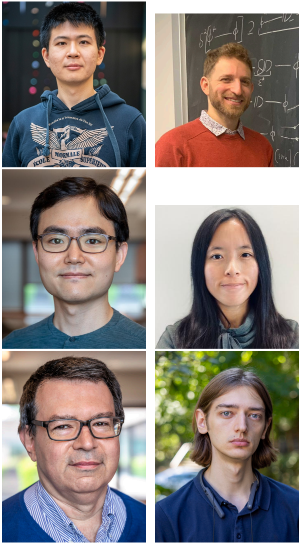
This talk is based on:
2507.21007 **HL, Zheng**
2511.08560 **Cho, Gabai, HL, Yeh, Zheng**

2603.04522 **Klebanov, HL, Meshcheriakov**



see also:

[Marchesini & Onofri '80]

[Gross & Klebanov '90]

[Maldacena '05]

[Cho, Gabai, Sandor, Yin, 2410.04262]

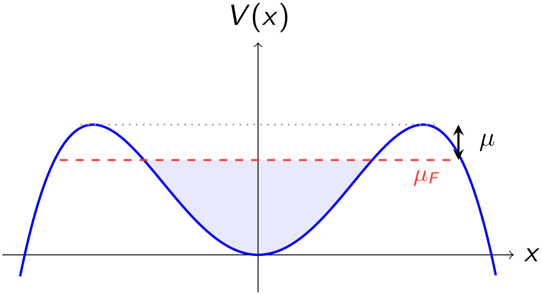
Matrix quantum mechanics

- Simplest toy model of a large N gauge theory:

$$L[X,A] = N\,\mathrm{Tr}\left[\frac{1}{2}(D_t X)^2 - V(X)\right], \qquad D_t X = \dot{X} + \mathrm{i}[A_0, X].$$

- Introduced almost 50 years ago. [BIPZ '78]

Potential near criticality



- Singlet sector reduces to N free fermions.
- As $g \to g_c$, $\mu = V_{\max} - \mu_F \to 0$.
- g_c is domain of analyticity; typical planar diagrams are big $\Rightarrow$ closed strings. Ripples in the Fermi sea.

Non-singlet sector

Much less is understood about this sector.

- Hilbert-space language: consider states that transform in some non-trivial irrep R of $U(N)$.
- Path-integral language: insert a Wilson line in representation R,

$$U_R(t_2, t_1) = \mathcal{P} \exp \left(\mathrm{i} \int_{t_1}^{t_2} dt \, A_0^a(t) \, T_R^a \right) .$$

- like in AdS/CFT, these non-singlet states should contain open strings, with endpoints in "the UV."

Why now?

New tools:

- DOZZ formula and a sharper understanding of Liouville theory,
- matrix bootstrap,
- better numerics.

New questions:

- analytic structure of thermal two-point functions,
- finite temperature and the BKT transition,
- what non-singlets know about folded strings.

[work in progress with Dodelson]

The matrix bootstrap

Bootstrapping matrix models using positivity + equations of motion.

1. Heisenberg eom: $\langle [H, O] \rangle = 0$
2. Hilbert space positivity: $\langle E | \operatorname{tr} \mathcal{O}^\dagger \mathcal{O} | E \rangle \geq 0$
3. Ground state positivity:

$$\mathcal{N}_{ij} = \langle \operatorname{tr} O_i^\dagger [H, O_j] \rangle_{\text{gs}} \succeq 0$$

4. Large N factorization $\langle \operatorname{tr} O_i \operatorname{tr} O_j \rangle = \langle \operatorname{tr} O_i \rangle \langle \operatorname{tr} O_j \rangle$.

[Anderson, Kruczenski; HL; Han, Hartnoll, Kruthoff; Kazakov, Zheng; ...]

Time-dependent bootstrap

Consider the ground state 2-pt function in Euclidean signature:

$$\mathcal{M}_{ij}(\tau) = \langle\Omega|\,\bar{\mathcal{O}}_i(\tau)\mathcal{O}_j(0)\,|\Omega\rangle$$

[CGLYZ]: bootstrap problem that gives rigorous bounds on $\mathcal{M}(\tau)$ for any desired τ, subject to:

- reflection positivity $\mathcal{M}(\tau) \succeq 0$,
- time-translation invariance $D^\dagger\mathcal{M} - \mathcal{M}D$,
- Heisenberg equations of motion $(D^\dagger + \partial_\tau)D = 0$.
- Ground state positivity

1-Matrix Quantum Mechanics

N^2 non-relativistic particles arranged in a matrix.

$$\mathrm{i}[X_{ij}, P_{kl}] = \delta_{il}\delta_{jk}$$

Hamiltonian:

$$H = N\left(\frac{1}{2}\,\mathrm{Tr}\,P^2 + \frac{m^2}{2}\,\mathrm{Tr}\,X^2 + \frac{g}{4}\,\mathrm{Tr}\,X^4\right)$$

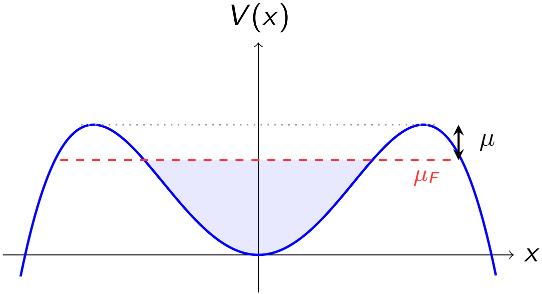
U(N) gauge constraint:

$$J_{ik} = \mathrm{i}(X_{ij}P_{jk} - P_{ij}X_{jk}) + N\delta_{ik} = 0$$

Known as $c = 1$ or $\hat{c} = 1$ matrix model (in the double scaling limit).
[BIPZ '78; for a review, see Klebanov hep-th/9108019]

Potential near criticality

The singlet sector reduces to N free fermions.Interested in the near-critical potential:



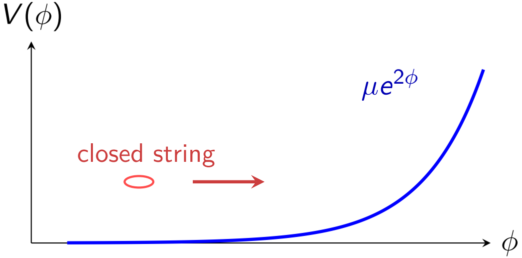
As $g \to g_c$, the Fermi level approaches the top of the barrier:

$$\mu = V_{\text{max}} - \mu_F \to 0$$

2D Liouville String

As $\mu \to 0$ singlet excitations are closed strings in a 2D target space:

$$I = \int \mathrm{d}^2\sigma \sqrt{\det g} \left[\frac{1}{4\pi} g^{ab} \left(\partial_a X^0 \partial_b X^0 + \partial_a \phi \partial_b \phi \right) + \frac{1}{4\pi} Q \phi R(g) - \mu e^{2\phi} \right]$$



Results from the time-dependent bootstrap

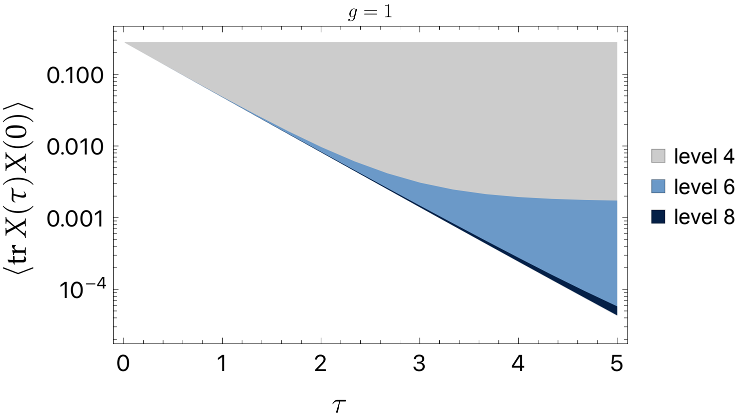
More interesting than the singlet excitations are the adjoint excitations. We studied the ground state correlator

$$\mathcal{M}(\tau) = \frac{1}{N} \langle\Omega| \operatorname{tr}[X(\tau)\ U(\tau,0)\ X(0)\ U(0,\tau)] |\Omega\rangle$$

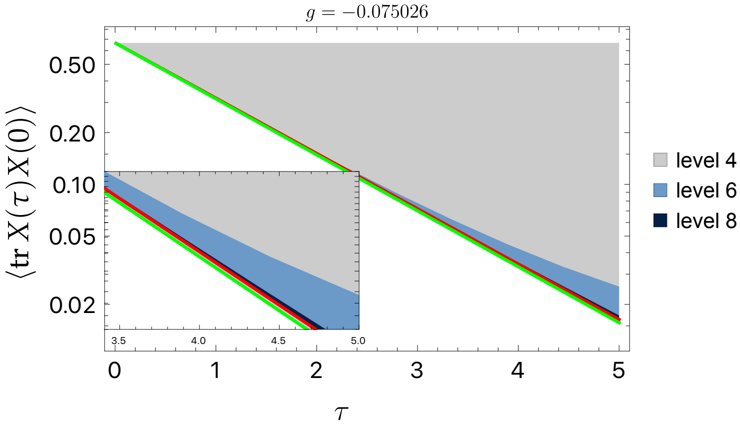
where $U(\tau_2,\tau_1) = \mathcal{P}\exp\Big(\mathrm{i}\int_{\tau_1}^{\tau_2} A_0\, dt'\Big)$ is the fundamental Wilson line.

The adjoint Wilson line $U \otimes U^\dagger$ connecting the two X insertions ensures gauge invariance and projects onto the adjoint sector.

Results from the time-dependent bootstrap

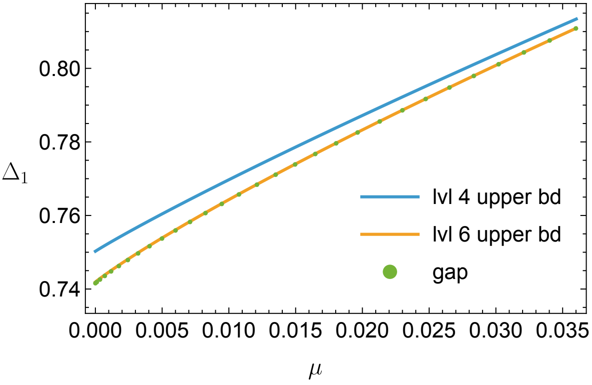


Adjoint gap near criticality



Adjoint gap near critical point. Red curve is computed using [Marchesini and Onofri '80]. Green is the analytical (level 4) bound.

Bound on the adjoint gap



Rigorous lower bound which rapidly converges.

A surprise

	1-param fit	2-param fit	MO	lvl 16 bd
Δ_1	0.74436	0.74199	0.74158	0.741573662448591
Δ_2	1.30914	1.30906	1.26130	1.26122
Δ_3	1.71290	1.68859	1.68218	1.68192**

It was suggested [Gross, Klebanov '90] that the adjoint gap *diverges* as $\mu \to 0$.

Instead, precision numerics shows the gap **stays finite**: $\Delta_1 \approx 0.7416$.

Question: what is the limiting adjoint spectrum as $g \to g_c$?
[with Igor Klebanov and Pavel Meshcheriakov]

The Marchesini-Onofri Equation

MQM review

Let us consider Schrodinger's equation for MQM. We work in position basis. This means

$$\Psi(X) = \Psi(\Lambda, \Omega), \quad X = \Omega\Lambda\Omega^{-1}, \quad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$$

In the gauge $X = \Omega\Lambda\Omega^{-1}$, S_N gauge redundancy that permutes the eigenvalues $\lambda_i \Rightarrow \Psi$ should be S_N-invariant. Further define

$$\Psi = \Delta^{-1}\psi(\Lambda, \Omega), \quad \Delta(\Lambda) = \prod_{i<j}(\lambda_i - \lambda_j)$$

Then ψ is anti-symmetric under swapping any $\lambda_i \leftrightarrow \lambda_j$.

MQM review

Acting on ψ, the Hamiltonian

$$H = \sum_{i=1}^{N} \left[-\frac{1}{2} \frac{\partial^2}{\partial \lambda_i^2} + V(\lambda_i) \right] + \frac{1}{2} \sum_{i \neq j} \frac{R_{ij} R_{ji}}{(\lambda_i - \lambda_j)^2},$$

where R_{ij} are the $U(N)$ generators:

$$R_{ij} = \sum_a \Omega_{ia} \frac{\partial}{\partial \Omega_{ja}}.$$

MQM review

For a $U(N)$ singlet, $\psi(\Lambda, \Omega) = \psi(\Lambda)$. Then,

$$H = \sum_{i=1}^{N} \left[-\frac{1}{2} \frac{\partial^2}{\partial \lambda_i^2} + V(\lambda_i) \right].$$

Furthermore ψ is restricted to be anti-symmetric under $\lambda_i \Rightarrow N$ independent fermions.

Adjoint wavefunction ansatz

Let $|\Omega\rangle$ be the singlet ground state.

An **adjoint state** is obtained by acting with a matrix element of $w(X)$:

$$|\Psi_{ab}\rangle \;=\; w(X)_{ab}\,|\Omega\rangle$$

This translates to a wavefunction:

$$\psi_{ab}(\Lambda,\Omega) \;=\; \sum_i \Omega_{ai} w(\lambda_i)\Omega^\dagger_{ib}\,\psi_0(\lambda_1,\ldots,\lambda_N)$$

We can work out the interaction term using $R_{kl} = \Omega_{km}\partial_{\Omega,lm}$,

$$R_{kl}R_{lk}(\Omega_{ai}\Omega^\dagger_{ib})w(\lambda_i) = (\Omega_{ak}\Omega^\dagger_{kb} - \Omega_{al}\Omega^\dagger_{lb})[w(\lambda_k) - w(\lambda_l)]$$

Marchesini-Onofri equation

The "radial" Schrodinger equation for the adjoint sector (at large N) is

$$E \sum_i w(\lambda_i) \Omega_{ai} \Omega^\dagger_{ib} \, \psi_0 \approx \sum_i \left[E_0 w(\lambda_i) - \sum_{i \neq j} \frac{w(\lambda_i) - w(\lambda_j)}{(\lambda_i - \lambda_j)^2} \right] \Omega_{ai} \Omega^\dagger_{ib} \, \psi_0$$

At large N: $\sum \to \int$ against singlet density of eigenvalues $\rho(\lambda)$

$$\boxed{\Delta_n \, w_n(\lambda) = \fint_{x_1}^{x_2} \mathrm{d}\lambda' \, \rho(\lambda') \, \frac{w_n(\lambda) - w_n(\lambda')}{(\lambda - \lambda')^2}}, \quad \rho(\lambda) = \frac{1}{\pi} \sqrt{2[\mu_F - V(\lambda)]}$$

This eigenvalue equation gives the *adjoint spectrum* $\{\Delta_n, w_n(\lambda)\}$.

MO equation

Define the rescaled eigenfunction

$$\phi_n(x) \;=\; \sqrt{2(\mu_F - V(x))}\, w_n(x), \qquad \phi_n(x_{1,2}) = 0.$$

Then the MO equation becomes

$$\Delta_n\, \phi_n(x) = \underbrace{\eta(\mu, x)\, \phi_n(x)}_{\text{potential}} \;-\; \underbrace{\sqrt{2(\mu_F - V(x))} \int_{x_1}^{x_2} \frac{\mathrm{d}y}{\pi} \frac{\phi_n(y)}{(x-y)^2}}_{\text{``kinetic''}}$$

where the **MO potential** is

$$\eta(\mu, x) \;\equiv\; \int_{x_1}^{x_2} \frac{\mathrm{d}y}{\pi} \frac{\sqrt{2(\mu_F - V(y))}}{(x-y)^2}$$

MO equation at criticality

For a particle at the Fermi energy,

$$\mathrm{d}\tau = \frac{\mathrm{d}x}{\dot{x}}, \qquad \dot{x} = \sqrt{2(\mu_F - V(x))} = \pi\rho(x),$$

so $\mathrm{d}\tau = \frac{\mathrm{d}x}{\pi\rho(x)}$ is just the classical flight time (up to normalization).

At the critical coupling $g = g_c$, the effective Hamiltonian *separates*:

$$\mathcal{H}_{\mathrm{quartic}}(p,\tau) = p \coth \frac{\pi p}{\sqrt{2}} + \frac{2}{\pi}\left(\tau \tanh \frac{\tau}{\sqrt{2}} - \sqrt{2}\right)$$

For sufficiently excited states:

$$\mathcal{H}_{\mathrm{quartic}}(p,\tau) \approx |p| + \frac{2}{\pi}\left(|\tau| - \sqrt{2}\right)$$

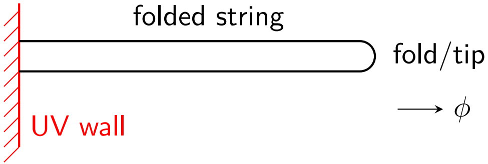
massless particle in a **linear potential.**

Folded Strings and Regge Trajectories

Review: folded strings in 2D string theory

2D string theory: target space has time X^0 and Liouville direction ϕ.

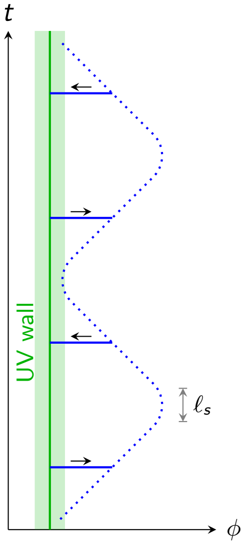
A folded open string extends from a "UV wall" at $\phi = 0$ into the bulk. The fold acts as a massless particle. [Bardeen, Bars, Hanson, Peccei '76; Maldacena '05]



String tension $= \frac{1}{2\pi\alpha'}$. Two segments of length $|\phi|$ give:

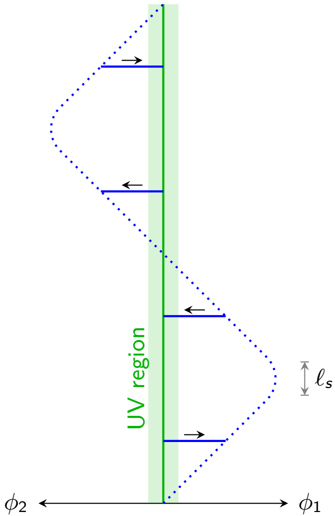
$$E = |p_\phi| + \frac{1}{\pi\alpha'}|\phi| + \text{const}$$

Oscillating folded string



The folded string bounces off the UV wall at $\phi = 0$.

Oscillating folded string



Quartic potential: two Liouville regions glued at $\phi = 0$. The $\mathbb{Z}_2$ symmetry acts as $\phi \to -\phi$.

Classical solution: the zig-zag

In static gauge $X^0 = \tau$, set $\mu = 0$ (far from the Liouville wall). The classical worldsheet solution is [Maldacena '05]:

$$\phi(\tau, \sigma) = \phi_0' - Q\alpha' \log \left(\cosh \frac{\tau}{\alpha' Q} + \cosh \frac{\sigma}{\alpha' Q} \right)$$

The tip ($\sigma = 0$) traces out a zig-zag:

$$\phi_{\text{tip}}(\tau) = \phi_0 - 2Q\alpha' \log \cosh \frac{\tau}{\alpha' Q}$$

Most of the time the tip moves at speed of light. Near the turning point it decelerates over a few string lengths $\ell_s \sim \sqrt{\alpha'}$.

Effect of the linear dilaton

The 2D string theory has a linear dilaton background with slope $Q = 2/\sqrt{\alpha'}$.

The spacetime energy of the folded string (with UV cutoff at ϕ_c) is [Maldacena '05]:

$$E = \frac{\phi - \phi_c}{\pi\alpha'} - \frac{Q}{\pi}\log\Big(1 - v^2\Big)$$

Using $P = \frac{2Q}{\pi}\,\text{arctanh}\, v$, the Hamiltonian becomes:

$$\boxed{E = \frac{\phi - \phi_c}{\pi\alpha'} + \frac{2Q}{\pi}\log\cosh\frac{\pi P}{2Q}}$$

From ultra-relativistic to non-relativistic

The kinetic term $T(P) = \frac{2Q}{\pi} \log\cosh \frac{\pi P}{2Q}$:

- large $|P|$: $T(P) \approx |P|$ (ultra-relativistic)
- small $|P|$: $T(P) \approx \frac{P^2}{2M}$, $M = \frac{2Q}{\pi} \sim 1/\ell_s$

The linear dilaton endows the fold with an **effective mass**.

Compare with the MO Hamiltonian $p \coth \frac{\pi p}{\sqrt{2}}$, which has the same qualitative structure: $\approx |p|$ at large p, $\approx \frac{p^2}{2M_{\text{fold}}}$ at small p with $M_{\text{fold}} = \frac{3Q}{4\pi}$.

Bohr–Sommerfeld quantization

At criticality: $\mathcal{H} \approx |p| + \frac{|\phi|}{\pi\alpha'}$ (relativistic particle in linear potential).

The classical orbit in phase space:

- turning point at $\phi_{\mathrm{m}} = \pi\alpha'\Delta$ (where $p = 0$)
- maximum momentum $p_{\max} = \Delta$ (at $\phi = 0$)

Phase space integral:

$$\underbrace{\tfrac{1}{2}\,\phi_{\mathrm{m}}\cdot\Delta}_{\oint p d\phi} = \frac{\pi\alpha'\Delta^2}{2} = \pi n \qquad \Longrightarrow \qquad \boxed{\Delta^2 = \frac{2n}{\alpha'}}$$

Regge trajectories: quartic potential

Semiclassical (Bohr-Sommerfeld) quantization of $\mathcal{H} \approx |p| + \frac{2}{\pi}|\tau|$ gives:

$$\boxed{\Delta_n^{\mathrm{Regge}} \approx \sqrt{2n + \frac{2}{3}} - \frac{2\sqrt{2}}{\pi}}$$

The $\sim \sqrt{n}$ growth is the hallmark of a **Regge trajectory**:

$\Delta^2 \sim n/\alpha'$ with $\alpha' = \frac{1}{2}$.

Valid for $n \lesssim n_{\max} \approx \frac{1}{4\pi^2} \log^2 \mu$.

The $\mathbb{Z}_2$ symmetry of the quartic potential gives **two** Regge trajectories (odd and even n).

Universality: cubic potential

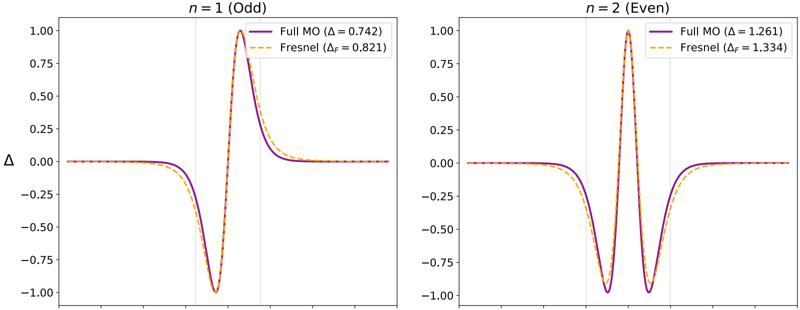
The cubic potential $V = \mathrm{tr}\left(\frac{1}{2}X^2 + gX^3\right)$ generates triangulated random surfaces. Near criticality, it corresponds to 2D string theory with $\alpha' = 1$.

No $\mathbb{Z}_2$ symmetry $\Rightarrow$ a **single** Regge trajectory:

$$\boxed{\Delta_n^{\text{cubic}} \approx \sqrt{2n + \tfrac{5}{6}} - \frac{3}{\pi}}$$

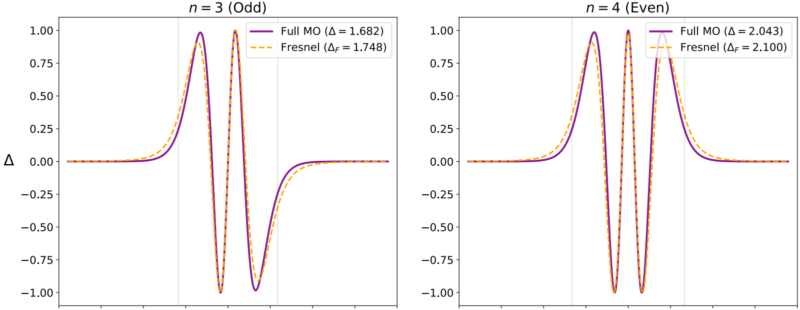
The Regge slope $\Delta^2 \sim 2n/\alpha'$ is **universal**: independent of the choice of matrix potential (quartic, cubic, double-well, ...), up to subleading corrections.

Eigenfunctions



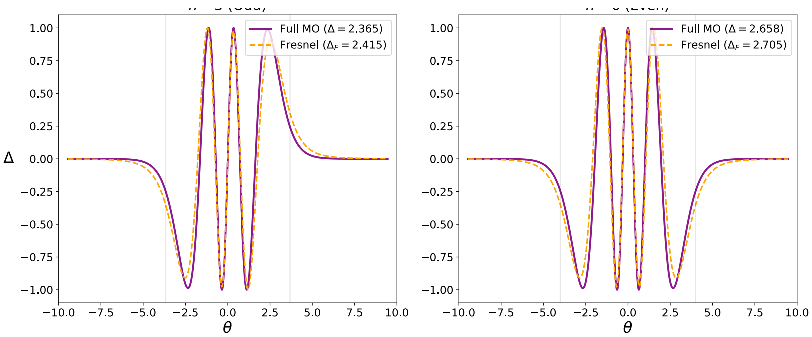
MO eigenfunctions at $\mu = 0$ (purple) vs. Fresnel approximation (orange, dashed).

Eigenfunctions



Odd/even wavefunctions under the quartic $\mathbb{Z}_2$ symmetry. Agreement improves with excitation number.

Eigenfunctions



Higher modes are already very well captured by the Fresnel approximation.

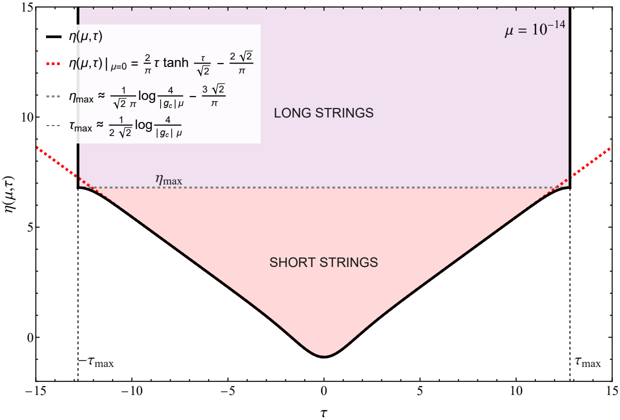
Numerical vs. analytic

	Quartic ($\alpha' = \frac{1}{2}$)			Cubic ($\alpha' = 1$)		
n	Δ_n^{num}	Δ_n^{an}	δ_n	Δ_n^{num}	Δ_n^{an}	δ_n
1	0.74157	0.73268	1.2%	0.72978	0.72832	0.2%
2	1.26122	1.25993	0.1%	1.24364	1.24355	0.007%
3	1.68192	1.68167	0.015%	1.65914	1.65914	0.0005%
4	2.04366	2.04360	0.003%	2.01716	2.01716	0.00005%
5	2.36569	2.36567	0.0007%	2.33647	2.33647	0.000007%

Even for $n = 1$ the Regge formula has $\sim 1\%$ relative error (quartic) or $\sim 0.2\%$ (cubic). Rapidly improves with n.

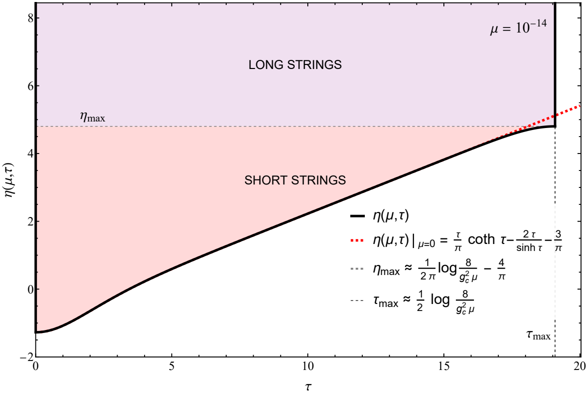
Slightly Away from Criticality

MO potential (quartic)



The effective potential at $\mu = 10^{-14}$. There is a clear separation between the UV region near $\tau = 0$ and the two Liouville walls at $\tau = \pm\tau_{\max}$.

MO potential



The effective potential has a UV wall at $\tau = 0$ and a single Liouville wall at $\tau = \tau_{\max}$.

The Liouville wall and $n_{\max}$

The plateau height $\eta_{\max}(\mu) \approx \frac{1}{\sqrt{2}\,\pi} \log \frac{1}{\mu}$ is the Liouville wall.

In the string theory picture: the Liouville potential $\mu\, e^{2\phi/\sqrt{\alpha'}}$ in the worldsheet action creates an exponential wall at

$$\phi_{\max} \;\sim\; \frac{\sqrt{\alpha'}}{2} \log \frac{1}{\mu}$$

For a string with fold tip at $\phi_{\rm tip}$: string energy $= \frac{1}{2\pi\alpha'} \times 2\phi_{\rm tip}$, so the turning point is

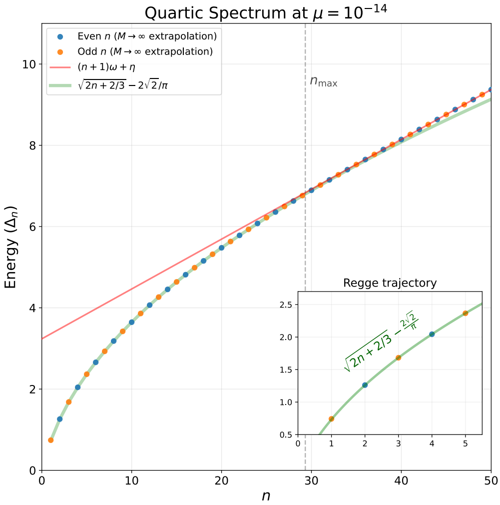
$$\phi_{\rm tip}(n) = \pi\alpha'\,\Delta_n$$

Crossover $\phi_{\rm tip}(n_{\max}) = \phi_{\max}$: $\quad \pi\alpha'\,\Delta_{n_{\max}} \sim \frac{\sqrt{\alpha'}}{2} \log \frac{1}{\mu}, \quad$ so

$$\Delta_{n_{\max}} \sim \frac{1}{2\pi\sqrt{\alpha'}} \log \frac{1}{\mu} \qquad \Longrightarrow \qquad n_{\max} \sim \frac{\log^2(1/\mu)}{8\pi^2\,\alpha'}$$

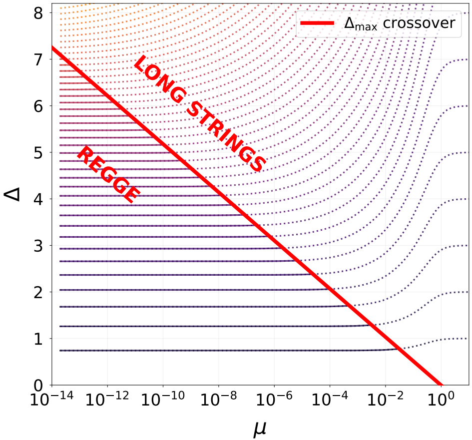
using $\Delta^2 \sim 2n/\alpha'$.

Quartic spectrum near criticality



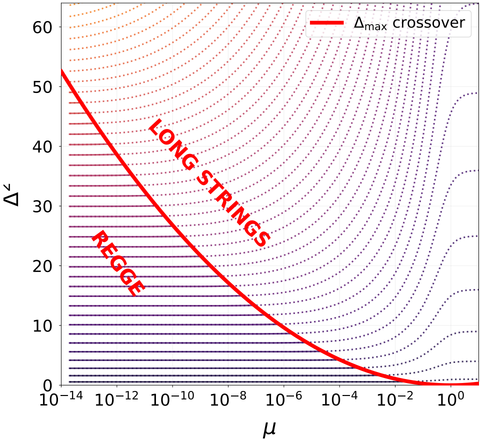
$\mu = 10^{-14}$. Blue/orange dots: $\mathbb{Z}_2$ even/odd states. Smooth interpolation between Regge ($n \lesssim n_{\max}$) and WKB ($n \gtrsim n_{\max}$) regimes.

Emergence of Regge behavior: weak coupling



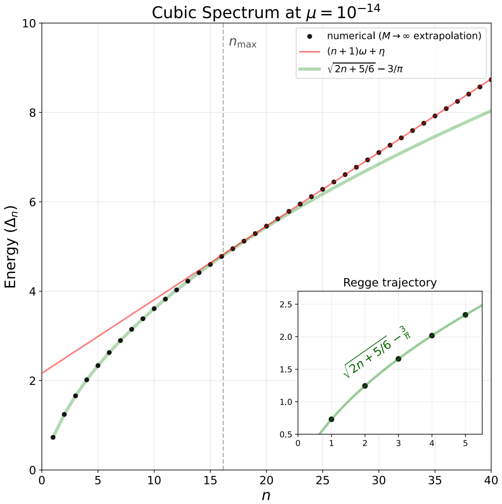
At large μ, Δ_n is evenly spaced ($\Delta_n \approx n\omega$): weakly coupled / long-string WKB regime throughout.

Emergence of Regge behavior: near criticality



For small μ, $\left(\Delta_n + \frac{2\sqrt{2}}{\pi}\right)^2$ becomes evenly spaced: Regge behavior $\Delta^2 \sim n/\alpha'$.
Need $\mu \lesssim 10^{-4}$ to resolve several Regge levels.

Cubic spectrum near criticality



$\mu = 10^{-14}$. Qualitatively similar to the quartic case but with no $\mathbb{Z}_2$ symmetry. Single Regge trajectory smoothly interpolates to the WKB regime.

Short strings vs. long strings

- **Short strings** ($n \lesssim n_{\max}$): tip of fold far from Liouville wall. Universal Regge behavior, $\Delta^2 \sim n/\alpha'$.
- **Long strings** ($n \gtrsim n_{\max}$): feel the Liouville potential. Particle in a box $\Rightarrow$ evenly spaced spectrum $\Delta_n \approx n\,\omega + \eta$.

The crossover is at $n_{\max} \sim \frac{1}{\alpha'} \cdot \frac{\log^2 \mu}{8\pi^2}$, when the tip of the fold reaches the Liouville wall:

$$4\pi\mu\, e^{2\phi_{\text{tip}}/\sqrt{\alpha'}} \sim 1$$

The "old" WKB result of [Gross, Klebanov '90] is the long-string regime. It correctly describes $n \gtrsim n_{\max}$, but not the Regge regime.

Implications for the BKT transition

Gross and Klebanov estimated the BKT critical temperature using the WKB approximation, finding a factor of 2 discrepancy with the continuum $T_{\mathrm{BKT}} = \frac{1}{4\pi\sqrt{\alpha'}}$.

Our finding: the WKB approximation is only valid for $n \gtrsim n_{\mathrm{max}}$. The equally spaced spectrum begins at

$$\Delta \approx \frac{1}{\sqrt{2}\pi} \log \frac{4}{|g_c|\mu}$$

which is **twice** the value assumed in [Gross, Klebanov '90]. This may resolve the discrepancy.

Discussion and future directions

- Despite being ~ 40 years old, precision numerics from the bootstrap led to new results about the $c = 1$ matrix model.
- Higher representations (dimension $\sim N^4$): energy levels are sums of MO eigenvalues $\Delta_i + \Delta_j$. String interpretation: multiple folds.
- Interpolation between short and long strings via Lüscher-like relation from the scattering phase shift? [Maldacena '05; Fidkowski]
- BKT transition and thermodynamics of MQM?
- Adjoint spectrum in BFSS?

Thanks!