Chiral Lattice Gauge Theories

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Based on [arXiv:2601.14359] and [arXiv:2604.06307] with **Zhiyao Lu** & **Shu-Heng Shao**

Lattice formulation of the Standard Model

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It is hard to put *chiral fermions* on the lattice due to

1. The Nielsen-Ninomiya theorem
2. Anomalies of chiral fermions

These obstacles make it hard, but not impossible, to find a lattice regularization of the Standard Model!

- The Standard Model is anomaly-free
- The goal is anomaly-free chiral lattice gauge theories

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- The Standard Model is anomaly-free
- The goal is anomaly-free chiral lattice gauge theories
- We address this in **1+1d** for *abelian* gauge theories
  See also [Berkowitz-Cherman-Jacobson 23, Thorngren-Preskill-Fidkowski 26]
- and report on some progress in **3+1d**

What is a chiral gauge theory?

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- In a chiral gauge theory, the left-moving and right-moving fermions transform differently under the gauge group
- The Lagrangian breaks reflection symmetry and is chiral
- Lattice regularization requires the realization of exact chiral symmetries of individual chiral fermions on the lattice
- The IR theory may have emergent parity symmetry
- The UV theory may have a hidden parity symmetry and be non-chiral!

Evading the Nielsen-Ninomiya no-go theorem

There is no discretization of the Dirac operator $D(p)$, such that

- **Locality**: $D(p)$ is a continuous and periodic function of p
- **Free fermions**: $D(p) \sim \gamma_\mu p^\mu$ for $|p| \ll 1/a$
- **No doublers**: $D(p)$ has poles only at $p = 0$
- **Exact chiral symmetry:** $\{\Gamma, D(p)\} = 0$

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Our strategy is to use duality/bosonization instead of directly discretizing the Dirac operators on the lattice:

1. Realizing exact anomalous chiral symmetries on the lattice
   Modified Villain: [Gross-Klebanov 90, Sulejmanpasic-Gattringer 19, Gorantla-Lam-Seiberg-Shao 21, Cheng-Seiberg 22, Fazza-Sulejmanpasic 22]
   3+1d: [Fidkowski-Xu-Zhang 25]
2. Gauging anomaly-free combination of chiral symmetries
   [SS 2023]

Goal today

1. Construct exactly solvable lattice Hamiltonians for chiral U(1) gauge theories in $\mathbf{1{+}1d}$

2. Construct a lattice model of bosons in $\mathbf{3{+}1d}$ with an exact chiral symmetry

1. **Chiral lattice gauge theories in 1+1d**
   - Bosonization and modified Villain model
   - I. The Schwinger model
   - II. The "34-50" model

2. **Exact lattice chiral symmetry in 3+1d**
   - 3+1d Villain model
   - Chiral symmetry and the continuum limit

Chiral lattice U(1) gauge theories in 1+1d

I. Realizing N free Dirac fermions and their $\mathrm{U}(1)^N_{\mathsf{L}} \times \mathrm{U}(1)^N_{\mathsf{R}}$ chiral symmetry on the lattice

$$\psi^{(1)}_{\mathsf{L}}, \psi^{(2)}_{\mathsf{L}}, \cdots, \psi^{(N)}_{\mathsf{L}} \quad , \quad \psi^{(1)}_{\mathsf{R}}, \psi^{(2)}_{\mathsf{R}}, \cdots, \psi^{(N)}_{\mathsf{R}}$$

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- No gravitational anomaly: $c_{\mathsf{L}} = c_{\mathsf{R}} = N$
- No gauge anomaly if:

$$\sum_{I=1}^{N} (n_{\mathsf{L}}^{(I)})^2 = \sum_{I=1}^{N} (n_{\mathsf{R}}^{(I)})^2$$

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- One popular option is the “34-50” model: $3^2 + 4^2 = 5^2 + 0^2$

Bosonization

A Dirac fermion in 1+1d is related by bosonization to the compact boson theory at radius $R = \frac{1}{\sqrt{2}}$:

$$\mathcal{L}_{\text{Dirac}} = i\psi_L^\dagger \partial_- \psi_{\text{L}} + i\psi_R^\dagger \partial_+ \psi_{\text{R}} \quad \Longleftrightarrow \quad \mathcal{L}_{\text{boson}} = \frac{1}{8\pi}\partial_\mu \varphi \partial^\mu \varphi$$

where $\varphi \sim \varphi + 2\pi$. The compact boson theory has $\mathrm{U}(1)_{\text{m}} \times \mathrm{U}(1)_{\text{w}}$ symmetry, which is related to chiral symmetries of fermions as

$$Q_{\text{L}} = Q_{\text{m}} + \frac{1}{2} Q_{\text{w}} \qquad Q_{\text{R}} = Q_{\text{m}} - \frac{1}{2} Q_{\text{w}}$$

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- In the "34-50" model, the anomaly-free conserved charge is:

$$Q = 3Q_{\mathrm{L}}^{(1)} + 4Q_{\mathrm{L}}^{(2)} + 5Q_{\mathrm{R}}^{(1)} = 8Q_{\mathrm{m}}^{(1)} + 4Q_{\mathrm{m}}^{(2)} - Q_{\mathrm{w}}^{(1)} + 2Q_{\mathrm{w}}^{(2)}$$

Modified Villain model: $\mathcal{L} = \frac{R^2}{4\pi}((\partial_0\varphi)^2 + (\partial_1\varphi)^2)$

The idea is to start from the non-compact boson theory

$$H = \sum_{j=1}^{L} \left(\frac{1}{2} p_j^2 + \frac{1}{2}(\phi_{j+1} - \phi_j)^2 \right) \quad \text{where: } [\phi_j, p_{j'}] = i\delta_{j,j'}$$

and gauge the shift symmetry $\phi_j \mapsto \phi_j + 2\pi R\mathbb{Z}$

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and gauge the shift symmetry $\phi_j \mapsto \phi_j + 2\pi R\mathbb{Z}$ to find

$$H_{\text{mV}} = \sum_{j=1}^{L} \left[\frac{1}{2R^2} p_j^2 + \frac{R^2}{2} \left(\frac{\phi_{j+1} - \phi_j}{2\pi} + \tilde{p}_{j,j+1} \right)^2 \right]$$

where $\tilde{p}_{j,j+1}$ are integer-valued gauge fields. They satisfy $[\tilde{\phi}_{j,j+1}, \tilde{p}_{j',j'+1}] = i\delta_{j,j'}$ and Gauss's law constraints:

$$e^{2\pi i p_j - i(\tilde{\phi}_{j,j+1} - \tilde{\phi}_{j-1,j})} = 1 : \qquad \begin{aligned} \phi_j &\sim \phi_j + 2\pi n_j \\ \tilde{p}_{j,j+1} &\sim \tilde{p}_{j,j+1} + n_j - n_{j+1} \end{aligned}$$

$$e^{2\pi i \tilde{p}_{j,j+1}} = 1 : \qquad \tilde{\phi}_{j,j+1} \sim \tilde{\phi}_{j,j+1} + 2\pi m_{j,j+1}$$

The chiral global symmetries of H_{mV}

$$Q_{\mathrm{m}} = \sum_{j=1}^{L} p_j \ , \qquad Q_{\mathrm{w}} = \sum_{j=1}^{L} \left(\tilde{p}_{j,j+1} + \frac{\phi_{j+1} - \phi_j}{2\pi} \right)$$

The momentum/winding charges are manifestly gauge invariant.
They are also quantized because of the gauge constraints:
$e^{2\pi i \tilde{p}_{j,j+1}} = 1$ and $e^{2\pi i p_j - i(\tilde{\phi}_{j,j+1} - \tilde{\phi}_{j-1,j})} = 1$

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- The anomaly can be seen from the Schwinger term:

$$\left[p_j \ , \ \left(\tilde{p}_{j',j'+1} + \frac{\phi_{j'+1} - \phi_{j'}}{2\pi} \right) \right] = \frac{i}{2\pi} \left(\delta_{j,j'} - \delta_{j,j'+1} \right)$$

which is a discretization of $\left[J^0_{\mathrm{m}}(t,x), J^0_{\mathrm{w}}(t,y) \right] = \frac{i}{2\pi} \partial_x \delta(x-y)$

Constructing chiral lattice gauge theories

We start with N flavors of the modified Villain theory

$$H_{\mathrm{mV}} = \sum_{I=1}^{N}\sum_{j=1}^{L}\left[\frac{1}{2R^2}\left(p_j^{(I)}\right)^2 + \frac{R^2}{2}\left(\tilde{p}_{j,j+1}^{(I)} + \frac{\phi_{j+1}^{(I)} - \phi_j^{(I)}}{2\pi}\right)^2\right]$$

with constraints $e^{2\pi i p_j^{(I)} - i(\tilde{\phi}_{j,j+1}^{(I)} - \tilde{\phi}_{j-1,j}^{(I)})} = 1$ and $e^{2\pi i \tilde{p}_{j,j+1}^{(I)}} = 1$

- Consider an anomaly-free U(1) chiral symmetry associated with the conserved charge

$$Q = \sum_{I=1}^{N}(n_{\mathrm{m}}^{(I)} Q_{\mathrm{m}}^{(I)} + n_{\mathrm{w}}^{(I)} Q_{\mathrm{w}}^{(I)})\ , \qquad \text{where: } \sum_{I=1}^{N} n_{\mathrm{m}}^{(I)} n_{\mathrm{w}}^{(I)} = 0$$

- Gauge this symmetry following [SS 2023]

Gauging $\mathrm{U}(1)_Q$

I. We couple the theory to background U(1) gauge fields $a_j \sim a_j + 2\pi$. This amounts to modifying Gauss's laws to

$$e^{2\pi i q_j^{(I)}} = e^{i n_{\mathrm{w}}^{(I)} a_j} \,, \qquad e^{2\pi i q_{j-1,j}^{(I)}} = e^{i n_{\mathrm{m}}^{(I)} a_j}$$

II. Making the gauge fields dynamical. We add conjugate variables (the electric field) E_j, where $[a_j, E_{j'}] = i\delta_{j,j'}$ and impose Gauss's law constraints

$$\sum_{I=1}^{N} \left(n_{\mathrm{m}}^{(I)} q_j^{(I)} + n_{\mathrm{w}}^{(I)} q_{j,j+1}^{(I)} \right) = E_j - E_{j+1} \,, \quad e^{2\pi i E_j} = 1$$

III. Add a kinetic term for the gauge fields to the Hamiltonian

$$\frac{e^2}{2} \sum_{j=1}^{L} \left(E_j + \frac{\theta}{2\pi} - \sum_{I=1}^{N} \frac{n_{\mathrm{w}}^{(I)} \phi_j^{(I)} + n_{\mathrm{m}}^{(I)} \tilde{\phi}_{j-1,j}^{(I)}}{2\pi} \right)^2$$

N=1 case: the "massless" Schwinger model

$$H_{\text{Schwinger}} = \sum_{j=1}^{L} \frac{e^2 a}{2} \left(E_j + \frac{\theta}{2\pi} - \frac{\phi_j}{2\pi} \right)^2 + \sum_{j=1}^{L} \frac{2\pi}{a} p_j^2 + \sum_{j=1}^{L} \frac{2\pi}{4a} \left(\tilde{p}_{j,j+1} + \frac{\phi_{j+1} - \phi_j}{2\pi} \right)^2$$

with constraints

$$e^{2\pi i p_j - i(\tilde{\phi}_{j,j+1} - \tilde{\phi}_{j-1,j})} = e^{i a_j} \,, \qquad e^{2\pi i \tilde{p}_{j,j+1}} = 1$$

$$\tilde{p}_{j,j+1} + E_{j+1} - E_j = 0 \,, \qquad e^{2\pi i E_j} = 1$$

N=1 case: the "massless" Schwinger model

$$H_{\text{Schwinger}} = \sum_{j=1}^{L} \frac{e^2 a}{2} \left(\cancel{E_j} + \frac{\theta}{2\pi} - \frac{\phi_j}{2\pi} \right)^2 + \sum_{j=1}^{L} \frac{2\pi}{a} p_j^2 + \sum_{j=1}^{L} \frac{2\pi}{4a} \left(\cancel{\tilde{p}_{j,j+1}} + \frac{\phi_{j+1} - \phi_j}{2\pi} \right)^2$$

with constraints

$$\cancel{e^{2\pi i p_j - i(\tilde{\phi}_{j,j+1} - \tilde{\phi}_{j-1,j})}} = e^{i a_j} \,, \qquad e^{2\pi i \tilde{p}_{j,j+1}} = e^{2\pi i (E_{j+1} - E_j)}$$

$$\tilde{p}_{j,j+1} + \cancel{E_{j+1} - E_j} = 0 \,, \qquad e^{2\pi i E_j} = 1$$

- We apply the unitary transformation $e^{2\pi i \sum_{j=1}^{L} E_j q_j}$

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- We apply the unitary transformation $e^{2\pi i \sum_{j=1}^{L} E_j q_j}$ to get

$$H_{\text{Schwinger}} = \sum_{j=1}^{L} \frac{e^2 a}{8\pi^2}\phi_j^2 + \sum_{j=1}^{L} \frac{2\pi}{a} p_j^2 + \sum_{j=1}^{L} \frac{2\pi}{4a}\left(\frac{\phi_{j+1}-\phi_j}{2\pi}\right)^2$$

which describes a non-compact boson of mass $M = \frac{e}{\sqrt{\pi}}$

The "34-50"/(8,4,-1,2) model

$$H_{8,4,-1,2} = \frac{e^2 a}{2} \sum_{j=1}^{L} \left(E_j + \frac{\theta}{2\pi} - \frac{8\tilde{\phi}^{(1)}_{j-1,j} + 4\tilde{\phi}^{(2)}_{j-1,j} - \phi^{(1)}_j + 2\phi^{(2)}_j}{2\pi} \right)^2$$

$$+ \frac{2\pi}{a} \sum_{j=1}^{L} \sum_{I=1}^{2} \left(\left(p^{(I)}_j \right)^2 + \frac{1}{4} \left(\tilde{p}^{(I)}_{j,j+1} + \frac{\phi^{(I)}_{j+1} - \phi^{(I)}_j}{2\pi} \right)^2 \right)$$

with constraints

$$e^{2\pi i q^{(1)}_j} = e^{-ia_j} \,, \qquad e^{2\pi i q^{(1)}_{j-1,j}} = e^{8ia_j}$$

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$$e^{2\pi i E_j} = 1 \,, \qquad 8q^{(1)}_j + 4q^{(2)}_j - q^{(1)}_{j,j+1} + 2q^{(2)}_{j,j+1} = E_j - E_{j+1}$$

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- To solve this, we perform a T-duality transformation $\mathcal{T}_{\mathcal{M}}$

The fermionic 34-50 model

The Hilbert space consists of two bosons $\phi_j^{(1)}, \phi_j^{(2)}$ and two Majorana fermions $\psi_j, \tilde{\psi}_j$ per site and one boson $\tilde{\phi}^{(2)}_{j,j+1}$ per link:

$$H_{3450} = \frac{e^2}{8\pi^2}\sum_{j=1}^{L}\left(\phi_j^{(1)}\right)^2 + \sum_{j=1}^{L}\left(\left(p_j^{(1)}\right)^2 + \left(-p_j^{(2)} + 2p_j^{(1)}\right)^2\right)$$
$$+ \sum_{j=1}^{L}\frac{1}{4}\left(-2\frac{(\mathrm{D}\phi^{(2)})_{j,j+1}}{2\pi} - 4p_j^{(2)} + 8p_{j+1}^{(1)} - \frac{(d\phi^{(1)})_{j,j+1}}{2\pi}\right)^2$$
$$+ \sum_{j=1}^{L}\frac{1}{4}\left(-\tilde{p}^{(2)}_{j,j+1} - \frac{\phi_{j+1}^{(2)} - \phi_j^{(2)}}{2\pi} + 4p_{j+1}^{(1)}\right)^2$$

with Gauss's law constraints

$$e^{2\pi i p_j^{(2)} - i(\tilde{\phi}^{(2)}_{j,j+1} - \tilde{\phi}^{(2)}_{j-1,j})} = i\tilde{\psi}_j\psi_j\ , \qquad i\psi_j\, e^{i\pi\tilde{p}^{(2)}_{j,j+1}}\,\tilde{\psi}_{j+1} = 1$$

The solution

The modified Villain theory has an exact $\mathrm{O}(N, N; \mathbb{Z})$ T-duality.
Using this, we solve the "34-50" model and determine its spectrum:
a massless Dirac fermion + a Schwinger boson of mass
$M = 5e/\sqrt{\pi}$ [Choi-Seiberg-**Seifnashri**-Zhang WIP]

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- In the continuum, the "34-50" model has a hidden reflection symmetry that is not manifest in the fermionic d.o.f.
- To get a chiral gauge theory, we need $N \geq 3$ Dirac fermions
- We can break the exact solvability by adding mass terms for the fermions (e.g., two-flavor massive Schwinger model [Cuomo-Dempsey-Katsevich-Klebanov-Kochergin-Pufu-Søgaard '26])

1. Chiral lattice gauge theories in 1+1d

   Bosonization and modified Villain model

   I. The Schwinger model

   II. The "34-50" model

2. Exact lattice chiral symmetry in 3+1d

   3+1d Villain model

   Chiral symmetry and the continuum limit

Villain model in 3+1d [Lu, SS, Shao '26]

Consider a boson ϕ_s at each site s where $[\phi_s, p_{s'}] = i\delta_{ss'}$ and also an integer-valued field w_ℓ at each link ℓ where $[b_\ell, w_{\ell'}] = i\delta_{\ell\ell'}$

$$H = \frac{1}{2\beta} \sum_{\text{sites } s} p_s^2 + \frac{\beta}{2} \sum_{\text{links } \ell} ((d\phi)_\ell + 2\pi w_\ell)^2 + \frac{\lambda}{2} \sum_{\text{plaquettes } p} (dw)_p^2$$

subject to gauge constraints:

$$e^{2\pi i p_s - i(\delta b)_s} = 1 : \qquad \phi_s \sim \phi_s + 2\pi n_s, \quad w_\ell \sim w_\ell - (dn)_\ell$$

$$e^{2\pi i w_\ell} = 1 : \qquad b_\ell \sim b_\ell + 2\pi m_\ell$$

where $(d\phi)_\ell = \phi_{s_1} - \phi_{s_2}$ for 

and $(\delta b)_s = \sum_{\ell \ni s} b_\ell$

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$$\begin{aligned} e^{2\pi i p_s - i(\delta b)_s} = 1 : \qquad & \phi_s \sim \phi_s + 2\pi n_s \\ & w_\ell \sim w_\ell - (dn)_\ell \\ e^{2\pi i w_\ell} = 1 : \qquad & b_\ell \sim b_\ell + 2\pi m_\ell \end{aligned}$$

where $(d\phi)_\ell = \phi_{s_1} - \phi_{s_2}$ for 

and $(\delta b)_s = \sum_{\ell \ni s} b_\ell$

- The λ-term imposes the flatness of the $\mathbb{Z}$ gauge fields w_ℓ at low energies and is necessary for a well-defined continuum limit

Global symmetries of the Villain Hamiltonian:

Vector: The vector symmetry $\mathrm{U}(1)_\mathsf{V} : \phi_s \to \phi_s + \alpha$ generated by

$$Q_\mathsf{V} = \sum_{\text{sites } s} p_s$$

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$$Q_\mathrm{V} = \sum_{\text{sites } s} p_s$$

Axial: The axial symmetry $\mathrm{U}(1)_\mathrm{A}$ is generated by

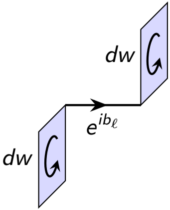
$$Q_\mathrm{A} = \int w \cup dw = \sum_{i,j,k} \epsilon_{ijk} \sum_{\text{sites } \vec{r}} w_i(\vec{r}) \left[w_j(\vec{r} + \hat{i}) - w_j(\vec{r} + \hat{i} + \hat{k}) \right]$$

The lattice cup product '$\cup$' is a discretization of the wedge product [Sulejmanpasic-Gattringer '19, DeMarco-Wen '21]. The formula for the axial charge is motivated by a similar axial symmetry discussed in [Fidkowski-Xu-Zhang '25, Thorngren-Preskill-Fidkowski '26], which acts discontinuously on the microscopic degrees of freedom

More on the $\mathrm{U}(1)_{\mathrm{V}} \times \mathrm{U}(1)_{\mathrm{A}}$ symmetry

Axial symmetry acts nontrivially on the short "axion string" e^{ib_ℓ} as

$$[Q_{\mathrm{A}}, e^{ib_\ell}] = \left((dw)_{\ell+(\frac{1}{2},\frac{1}{2},\frac{1}{2})} + (dw)_{\ell-(\frac{1}{2},\frac{1}{2},\frac{1}{2})}\right) e^{ib_\ell} \ ,$$



- The $\mathrm{U}(1)_{\mathrm{V}} \times \mathrm{U}(1)_{\mathrm{A}}$ symmetry has the V-V-A triangle anomaly, but there is no A-A-A and A-Grav-Grav anomaly
- This anomaly is different from that of a single Weyl fermion, but is realized by two Dirac fermions

Continuum limit : $\phi_s \mapsto \varphi(x)$ and $b_\ell \mapsto B_{\mu\nu}(x)$

The continuum limit of our lattice Hamiltonian is a free compact boson (i.e., an axion or a Goldstone):

$$\mathcal{L} = \frac{f^2}{2}\left(\partial_\mu \varphi - A^{\mathsf{V}}_\mu\right)^2 + \frac{i}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\varphi F^{\mathsf{V}}_{\mu\nu} F^{\mathsf{A}}_{\rho\sigma}\ , \qquad \text{for } \beta = f^2 a^2$$

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- $\mathrm{U}(1)_{\mathsf{A}}$ does not act faithfully in the IR and *transmutes* into a higher-form symmetry of compact boson [Seiberg-SS '25], which matches the UV anomaly [Cordova-Dumitrescu-Intriligator '18]

Is there a deformation to chiral fermions?

The compact boson theory $\mathcal{L} = \frac{f^2}{2}(\partial_\mu)^2$ has another UV realization via a Yukawa field theory:

$$\mathcal{L}_{\text{Yukawa}} = g\Phi^\dagger \psi^1 \psi^3 + g\Phi \psi^2 \psi^4 + \text{h.c.} + m^2|\Phi|^2 + \lambda|\Phi|^4$$

	ψ^1	ψ^2	ψ^3	ψ^4	Φ
Q_V	$+1$	-1	0	0	$+1$
Q_A	$+1$	$+1$	-1	-1	0

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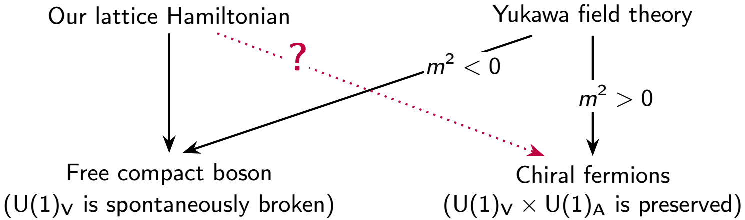

Summary & Outlook

- **1+1d:** Constructed exactly solvable chiral lattice $\mathrm{U}(1)$ gauge theories via gauging anomaly-free combinations of chiral symmetries
- **3+1d:** Realized exact $\mathsf{U}(1)_\mathsf{V} \times \mathsf{U}(1)_\mathsf{A}$ chiral symmetry and its anomaly in a concrete lattice Hamiltonian of bosons
- We also derived the anomalies of $\mathsf{U}(1)_\mathsf{V} \times \mathsf{U}(1)_\mathsf{A}$ from the structure of the generalized global symmetries obtained by gauging $\mathsf{U}(1)_\mathsf{V}$ and $\mathsf{U}(1)_\mathsf{A}$ [Lu, SS, Shao 2026]

- Generalization to **non-abelian** gauge groups such as SU(N)?
- U(1) chiral symmetry in 3+1d with U(1)-**grav-grav** anomaly?
- Is there a 3+1d lattice Hamiltonian that **does not spontaneously break** the chiral symmetry $\mathsf{U}(1)_\mathsf{V} \times \mathsf{U}(1)_\mathsf{A}$?

Thank You!

T-duality of $H_{\rm mV}$

The modified Villain Hamiltonian realizes an exact T-duality

$$\mathcal{T}: \quad \begin{array}{cc} \phi_j \mapsto \tilde{\phi}_{j,j+1} \ , & p_j \mapsto \tilde{p}_{j,j+1} + \dfrac{\phi_{j+1} - \phi_j}{2\pi} \\ \tilde{\phi}_{j,j+1} \mapsto \phi_{j+1} \ , & \tilde{p}_{j,j+1} \mapsto p_{j+1} - \dfrac{\tilde{\phi}_{j+1,j+2} - \tilde{\phi}_{j,j+1}}{2\pi} \end{array}$$

It implements T-duality and is gauge invariant since

$$\mathcal{T}: \quad \begin{array}{l} H_{\rm mV}(R) \mapsto H_{\rm mV}(R^{-1}) \\ q_j \mapsto q_{j,j+1} \ , \qquad q_{j,j+1} \mapsto q_{j+1} \end{array}$$

- $\mathcal{T}$ exchanges the momentum and winding charges as expected: $\mathcal{T} Q_{\rm m} \mathcal{T}^{-1} = Q_{\rm w}$ and $\mathcal{T} Q_{\rm w} \mathcal{T}^{-1} = Q_{\rm m}$
- $\mathcal{T}$ acts as a half-translation since $\mathcal{T}^2 = T_{\rm translation}$

Alternative presentation of the gauge theory

Applying the transformation $e^{-i\sum_{I,j}\left(\frac{n_{\mathrm{w}}^{(I)}}{2\pi}\phi_j^{(I)}+\frac{n_{\mathrm{m}}^{(I)}}{2\pi}\tilde{\phi}_{j-1,j}^{(I)}\right)a_j}$, the gauged Hamiltonian becomes

$$H_{\mathrm{gauged}} = \frac{e^2}{2}\sum_{j=1}^{L}\left(E_j+\frac{\theta}{2\pi}\right)^2 + \sum_{I=1}^{N}\sum_{j=1}^{L}\frac{1}{2R^2}\left(p_j^{(I)}+\frac{n_{\mathrm{w}}^{(I)}}{2\pi}a_j\right)^2 + \sum_{I=1}^{N}\sum_{j=1}^{L}\frac{R^2}{2}\left(\tilde{p}_{j,j+1}^{(I)}+\frac{n_{\mathrm{m}}^{(I)}}{2\pi}a_{j+1}+\frac{\phi_{j+1}^{(I)}-\phi_j^{(I)}}{2\pi}\right)^2$$

Gauss's law for U(1) gauging and the quantization of the electric fields change to

$$\sum_{I=1}^{N}\left(n_{\mathrm{m}}^{(I)}p_j^{(I)}+n_{\mathrm{w}}^{(I)}\left(\tilde{p}_{j,j+1}^{(I)}+\frac{\phi_{j+1}^{(I)}-\phi_j^{(I)}}{2\pi}\right)\right) = E_j - E_{j+1}$$

$$e^{2\pi i E_j} = \exp\left\{-i\sum_{I=1}^{N}\frac{n_{\mathrm{w}}^{(I)}}{2\pi}\phi_j^{(I)}+\frac{n_{\mathrm{m}}^{(I)}}{2\pi}\tilde{\phi}_{j-1,j}^{(I)}\right\}$$

Solving 1+1d chiral gauge theories using T-duality

In the continuum, we can solve these model by performing a T-duality that transforms the chiral charge Q into a non-chiral charge such as $Q_{\mathrm{w}}^{(1)}$. Doing such makes the charge non-chiral, but makes the coupling between different bosons chiral:

$$\int \frac{1}{8\pi}(\mathrm{d}\varphi^{(I)} + n_{\mathrm{m}}^{(I)} a) \wedge \star(\mathrm{d}\varphi^{(I)} + n_{\mathrm{m}}^{(I)} a) + \frac{n_{\mathrm{w}}^{(I)}}{2\pi}\varphi^{(I)}\mathrm{d}a \sim$$
$$\int G_{IJ}\mathrm{d}\varphi^{(I)} \wedge \star\mathrm{d}\varphi^{(J)} + B_{IJ}\mathrm{d}\varphi^{(I)} \wedge \mathrm{d}\varphi^{(J)} + \frac{n}{2\pi}\phi^{(1)}\mathrm{d}a$$

such that $n = \gcd(n_{\mathrm{m}}^{(I)}, n_{\mathrm{w}}^{(I)})$ and G_{IJ} and B_{IJ} are obtained by the T-duality transformation $\mathcal{M} \in \mathrm{O}(N, N; \mathbb{Z})$ satisfying

$$\mathcal{M}\begin{pmatrix}\vec{n}_{\mathrm{m}}\\ \vec{n}_{\mathrm{w}}\end{pmatrix} = \begin{pmatrix}0\\ \delta_{I,1}\end{pmatrix}$$

Solving the (8,4,-1,2) model using T-duality

$$\mathcal{M} = \begin{pmatrix} -1 & 2 & 8 & 4 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & -1 \end{pmatrix} : \begin{pmatrix} 8 \\ 4 \\ -1 \\ 2 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathcal{T}_{\mathcal{M}} Q \mathcal{T}_{\mathcal{M}}^{-1} = Q_{\mathrm{w}}^{(1)}$$

$$\mathcal{T}_{\mathcal{M}} : \quad \begin{aligned} \phi_j^{(1)} &\mapsto -\phi_j^{(1)} - 2\phi_j^{(2)} - 4\tilde{\phi}_{j-1,j}^{(2)} + 8\tilde{\phi}_{j,j+1}^{(1)} \\ \phi_j^{(2)} &\mapsto -\phi_j^{(2)} + 4\tilde{\phi}_{j,j+1}^{(1)} \\ \tilde{\phi}_{j,j+1}^{(1)} &\mapsto -\tilde{\phi}_{j,j+1}^{(1)} \\ \tilde{\phi}_{j,j+1}^{(2)} &\mapsto -\tilde{\phi}_{j,j+1}^{(2)} + 2\tilde{\phi}_{j,j+1}^{(1)} \\ p_j^{(1)} &\mapsto -p_j^{(1)} \\ p_j^{(2)} &\mapsto -p_j^{(2)} + 2p_j^{(1)} \\ \tilde{p}_{j,j+1}^{(1)} &\mapsto -\tilde{p}_{j,j+1}^{(1)} - 2\tilde{p}_{j,j+1}^{(2)} - 4q_j^{(2)} + 8q_{j+1}^{(1)} \\ \tilde{p}_{j,j+1}^{(2)} &\mapsto -\tilde{p}_{j,j+1}^{(2)} + 4q_{j+1}^{(1)} \end{aligned}$$