Schwinger model at $\theta = \pi$: solution of Coleman's puzzle

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Based on arXiv:2605.08042 with Gabriel Cuomo, Ross Dempsey, Andrei Katsevich, Igor Klebanov, Silviu Pufu and Benjamin Søgaard

The model

- 1+1D QED with 2 fundamental fermions and the θ - term; $\theta \sim \theta + 2\pi$

$$\mathscr{L} = \sum_{\alpha=1}^{2} \left(i\bar{\psi}^{\alpha}\hat{D}\psi_{\alpha} - m_{\alpha}\bar{\psi}^{\alpha}\psi_{\alpha} \right) - \frac{1}{4g^2}F_{\mu\nu}^2 - \frac{\theta}{4\pi}\varepsilon^{\mu\nu}F_{\mu\nu}$$

- Isospin symmetry SU(2)/$\mathbb{Z}_2 \cong$ SO(3) for $m_1 = m_2 = m \neq 0$

- Discrete symmetries: parity P and charge conjugation C at $\theta = 0, \pi$:

$$P: \ (A_\mu(t,x), \psi_\alpha(t,x)) \longrightarrow (\eta_{\mu\mu}A_\mu(t,-x), \gamma^0\psi_\alpha(t,-x))$$

$$C: \ (A_\mu(t,x), \psi_\alpha(t,x)) \longrightarrow (-A_\mu(t,x), \gamma^5\psi_\alpha^*(t,x))$$

- Not solvable, but there are various analytical methods at weak and strong coupling

- Numerical results: infinite MPS (see Anna's talk)

Spectrum at weak coupling $m \gg g$

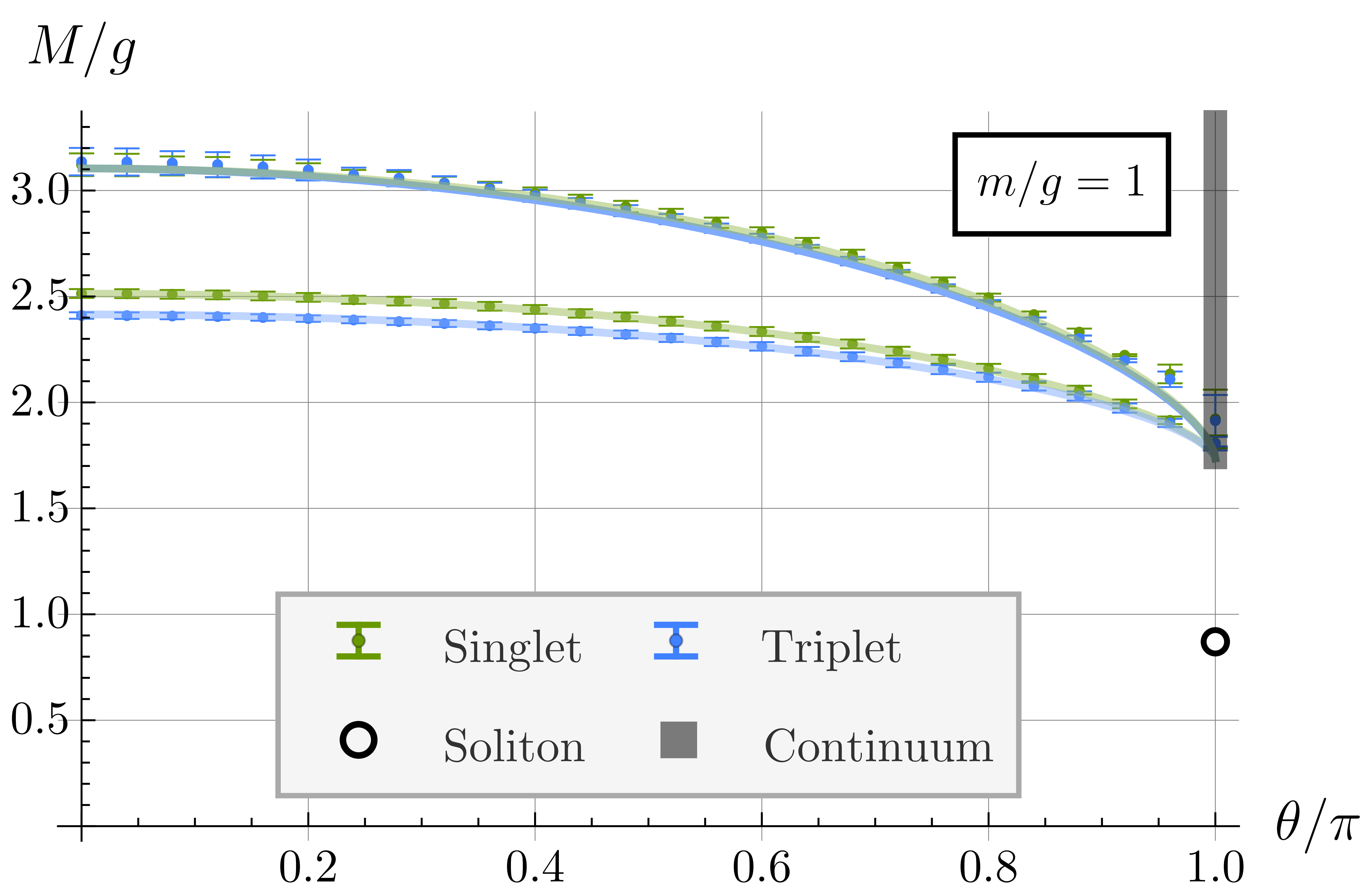
- θ - term $\cong$ background electric field $\frac{\theta g^2}{2\pi}$; Two vacua at $\theta = \pi$: C is broken semiclassically

- Effective fermion-antifermion Hamiltonian

$$H_{\text{NR}} = \frac{p^2}{m} + \frac{g^2}{2}\left(|x| - \frac{\theta}{\pi}x \right);$$

$$\Delta H_{\text{singlet}} = \frac{g^2}{2m^2}\delta(x);$$

$$M = 2m + E_{\text{NR}}; \quad E_{\text{NR}} \sim g^{4/3}m^{-1/3}$$



- The lightest particle π is parity-odd

- Half-asymptotic particles (solitons) at $\theta = \pi$ [Coleman'1976]—they interpolate between the vacua. Neutral states are in the continuum

Spectrum at strong coupling $g \gg m$

- Introduce bosonic periodic fields $\phi_\pm$ and integrate out A_μ; integrate out massive ϕ_+ at strong coupling:

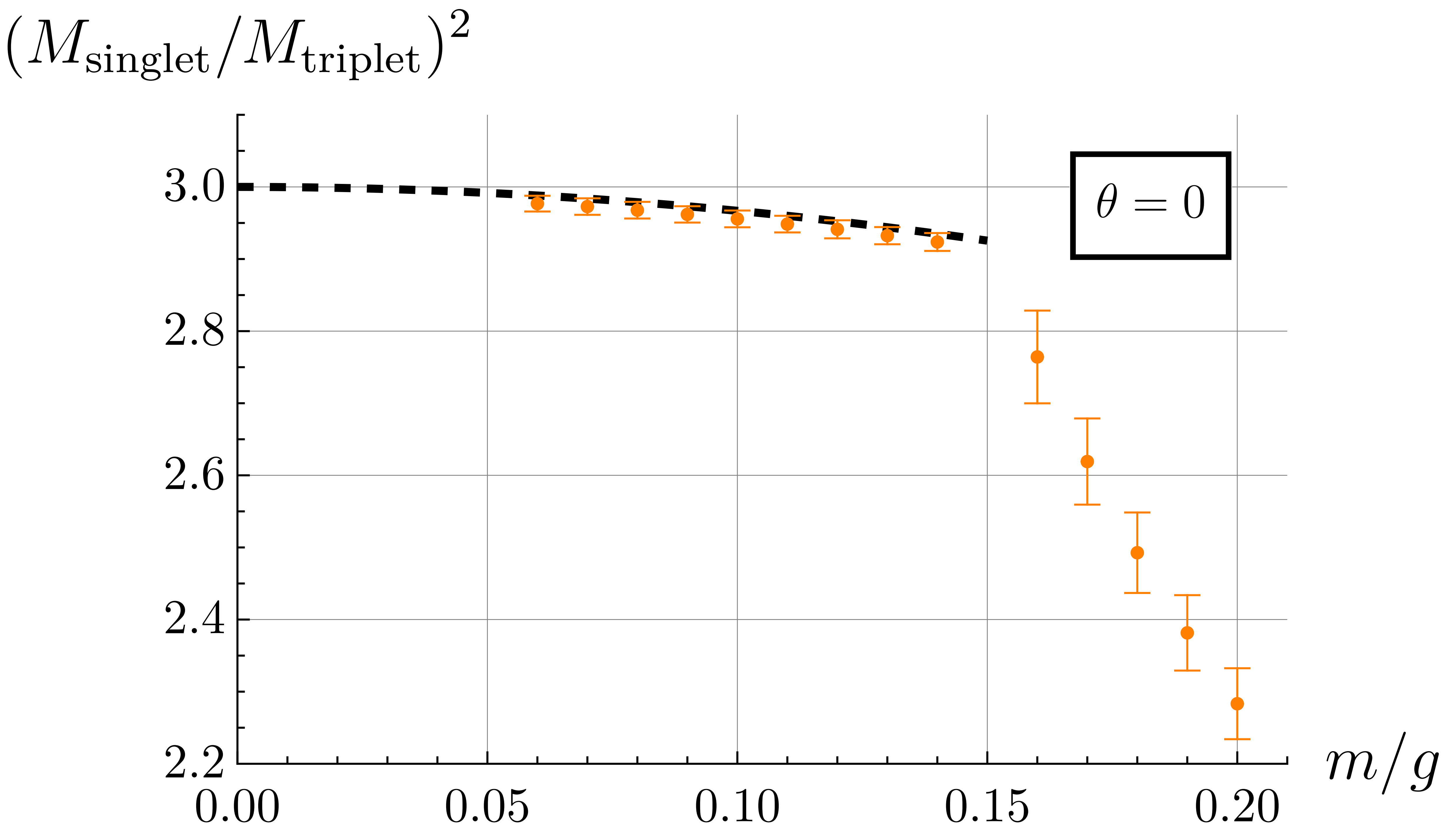
$$\mathscr{L}_{\text{EFT}} \simeq \frac{1}{2}\partial_\mu\phi_-\partial^\mu\phi_- + \cos\left(\frac{\theta}{2}\right)\frac{e^{\gamma_E}}{\pi}m\sqrt{\mu\,\mu_-}N_{\mu_-}\cos(\sqrt{2\pi}\phi_-), \quad \mu = \sqrt{\frac{2}{\pi}}g \text{ — mass of } \phi_+$$

- Integrable theory: triplet π and singet σ;

$$M_\pi \simeq 2.008\left(m\cos\frac{\theta}{2}\right)^{2/3} g^{1/3}, \;\; M_\sigma/M_\pi = \sqrt{3}$$

[Zamolodchikov'1995], [Smilga'1997]

- Subleading EFT terms $\sim \cos(\sqrt{8\pi}\phi_-)$ and $\sim (\partial\phi)^2$ are marginal, become leading at $\theta = \pi$

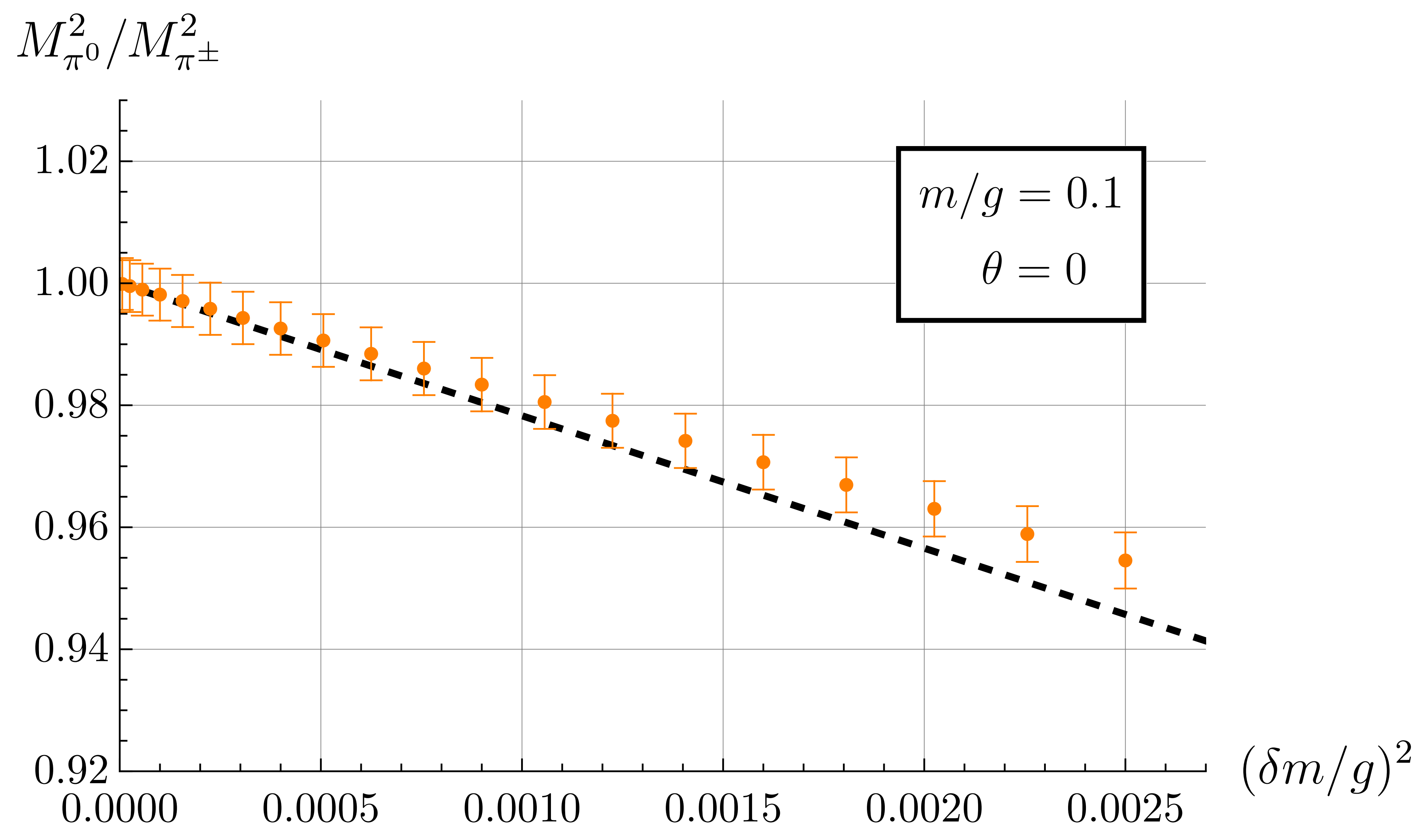


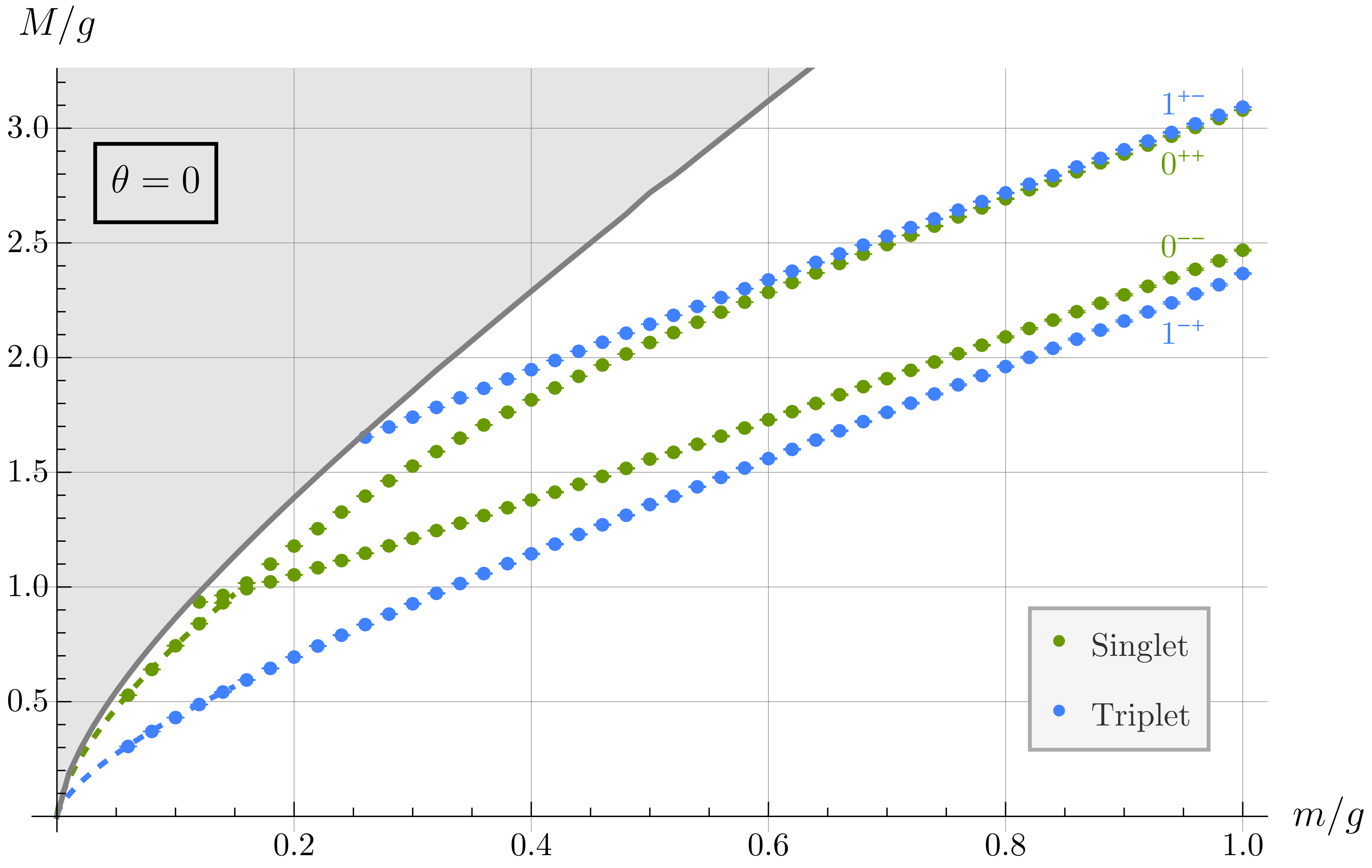
Coleman's puzzles of at $\theta = 0$

- Coleman formulated 3 puzzles in "More about the massive Schwinger model" [Coleman'1976]

- Puzzle 1: "Why are the lightest particles in the theory a degenerate isotriplet, even if one quark is 10 times heavier than the other?" Solution: SO(3) is preserved by leading order EFT: $\delta m = (m_1 - m_2)/2$; $m = (m_1 + m_2)/2$;

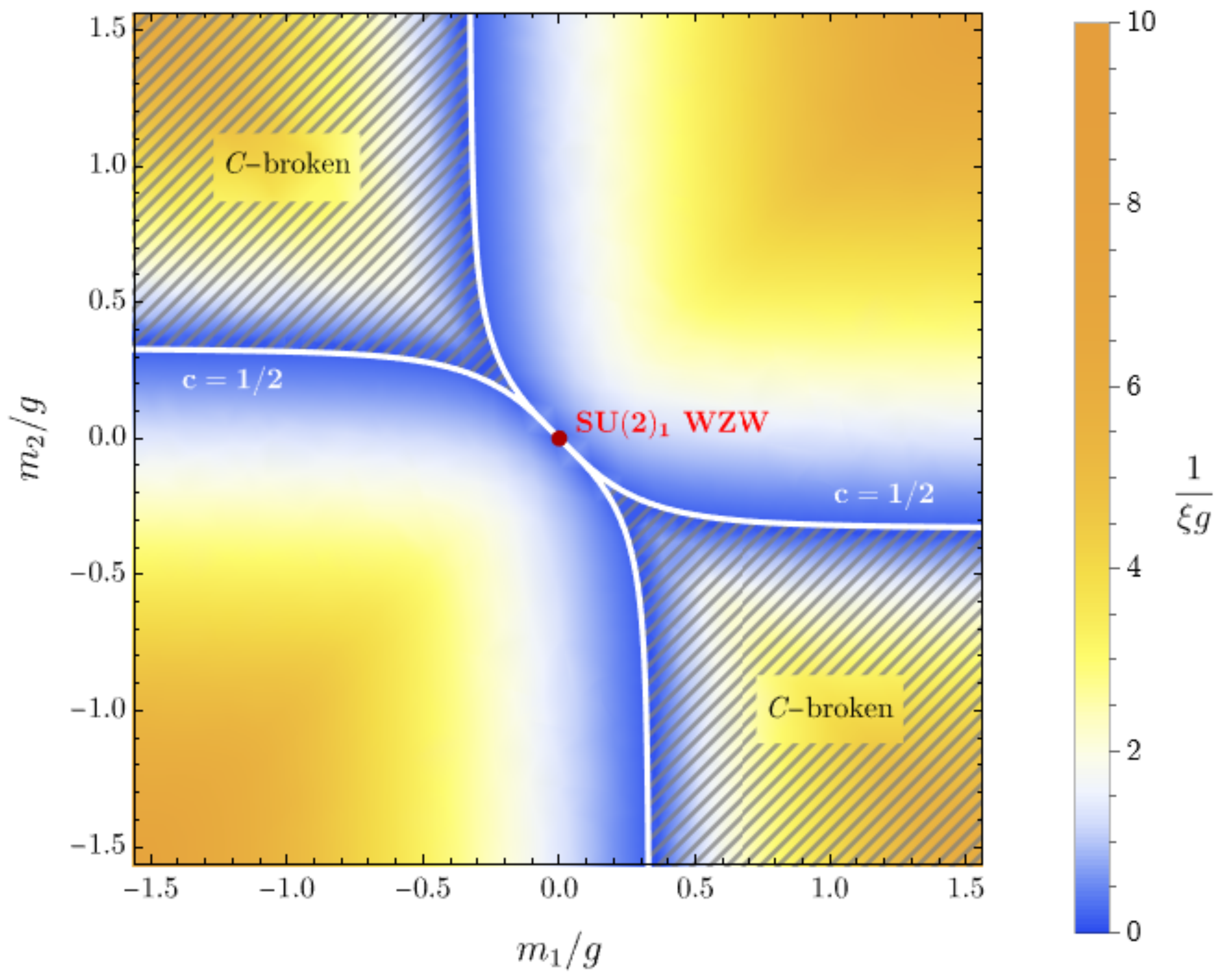
- Puzzle 2: "Why does the next lightest particle have $I^{PG} = 0^{++}$, rather than 0^{--}?" Solution: level-crossing at $m/g \sim 0.15$, σ is tetraquark-like





The puzzle at $\theta = \pi$

- $V(\phi_-) \sim -\cos(\sqrt{8\pi}\phi_-)$: minima at $\phi_- = 0$, $\phi_- = \sqrt{\pi/2}$. Solitons are doublets with $I = 1/2$ and (naively) zero gauge charge
- Mixed anomaly between $\theta \to \theta + 2\pi$ and SO(3) [Cordova et. al'2019]: solitons are in a projective representation
- Puzzle 3: "For $\theta = \pi$, how can an isodoublet quark and an isodoublet antiquark, carrying opposite electric charges, make an isodoublet bound state of electric charge zero?"
- No phase transition, always gapped at $m \neq 0$ [Dempsey et. al'2023]



Semiclassical analysis I

- The full potential with minimum over gauge field sectors [Komargodski et. al'2020]

$$V(\phi_-) = \frac{\mu^2}{2}\text{min}_{n\in\mathbb{Z}}\left(\phi_+ + \frac{\theta+2\pi n}{2\sqrt{2\pi}}\right)^2 - \frac{e^{\gamma_E}}{\pi} m\sqrt{\mu\mu_-}\, N_\mu \cos\left(\sqrt{2\pi}\phi_+\right) N_{\mu_-}\cos\left(\sqrt{2\pi}\phi_-\right);$$

$$(\phi_+,\phi_-) \sim (\phi_+ \pm \sqrt{\pi/2}, \phi_- \pm \sqrt{\pi/2}); \qquad C: \phi_\pm \to \mp\sqrt{\pi/2} - \phi_\pm$$

- $g \ll m$ vacua at $\theta = \pi$:

$$\begin{cases} \phi_- = 0 & \phi_+ = -\dfrac{2\pi\,\mu}{8\sqrt{2\pi}\,e^{\gamma_E} m} + \mathcal{O}\left(\mu^2/m^2\right); \\ \phi_- = \sqrt{\dfrac{\pi}{2}} & \phi_+ = -\sqrt{\dfrac{\pi}{2}} + \dfrac{2\pi\,\mu}{8\sqrt{2\pi}\,e^{\gamma_E} m} + \mathcal{O}\left(\mu^2/m^2\right). \end{cases}$$

- Isospin cartan and gauge charges: $Q_3 = \frac{1}{\sqrt{2\pi}}\int dx\,\partial_x\phi_-$, $Q_{\text{gauge}} = \sqrt{\frac{2}{\pi}}\int dx\,\partial_x\phi_+$:

$$Q_3 = \pm 1/2, \quad Q_{\text{gauge}} = 1 - \frac{\mu}{2e^{\gamma_E} m} + \mathcal{O}\left(g^2/m^2\right)$$

Semiclassical analysis II

- $g \gg m$ vacua at $\theta = \pi$:
$$\begin{cases} \phi_- = 0 & \phi_+ = -\sqrt{\frac{\pi}{8}} + \frac{\sqrt{2\pi} e^{\gamma_E} m}{\pi \mu} + \mathcal{O}\left(m^3/\mu^3\right); \\ \phi_- = \sqrt{\frac{\pi}{2}} & \phi_+ = -\sqrt{\frac{\pi}{8}} - \frac{\sqrt{2\pi} e^{\gamma_E} m}{\pi \mu} + \mathcal{O}\left(m^3/\mu^3\right). \end{cases}$$

- The charges: $Q_3 = \pm 1/2, \quad Q_{\text{gauge}} = \dfrac{4 e^{\gamma_E} m}{\sqrt{2\pi}\,\mu} + \mathcal{O}\left(m^3/\mu^3\right)$

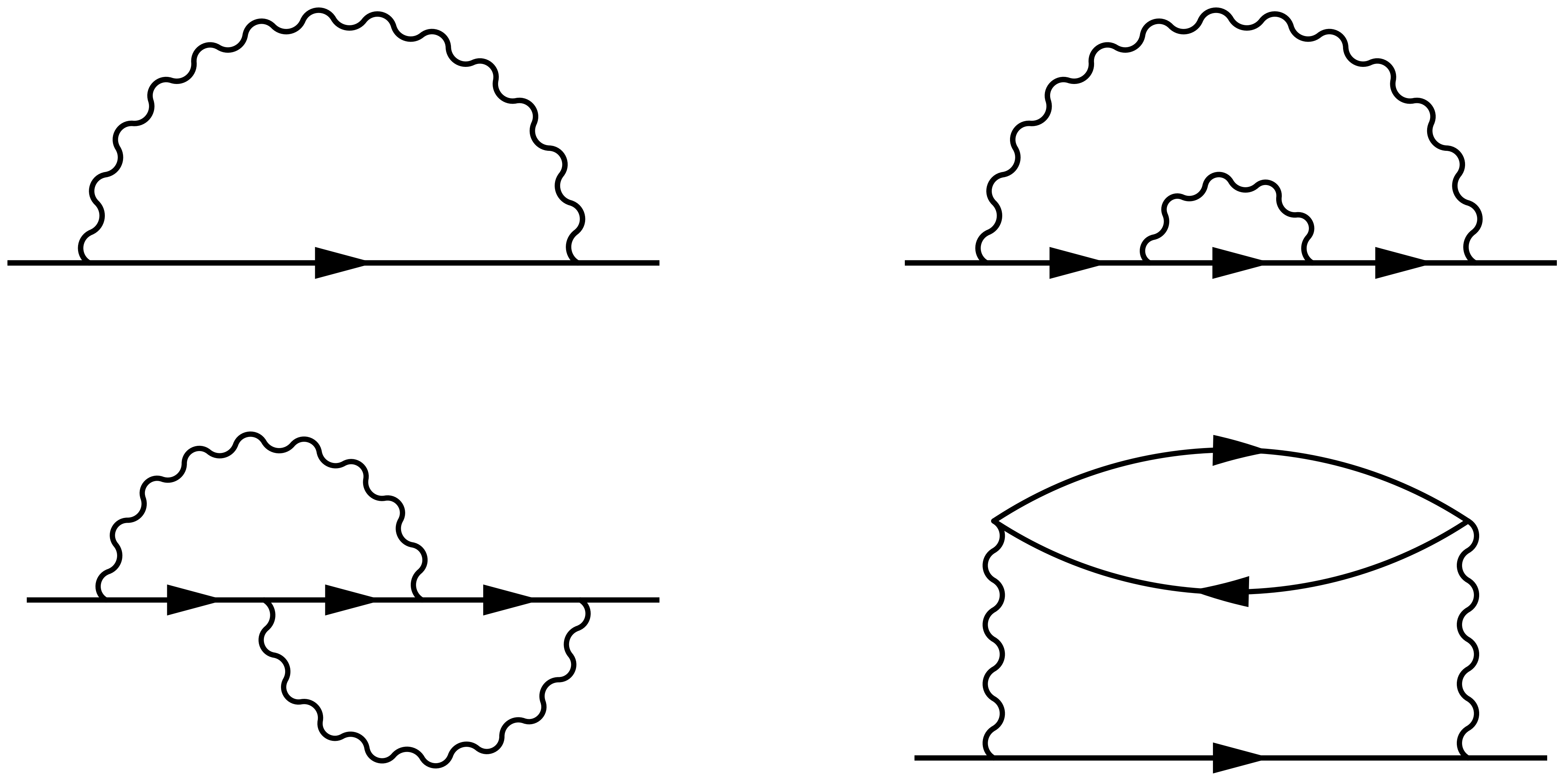
- Smooth interpolation between $Q_{\text{gauge}} = 1$ at $m \to \infty$ and $Q_{\text{gauge}} = 0$ at $m \to 0$

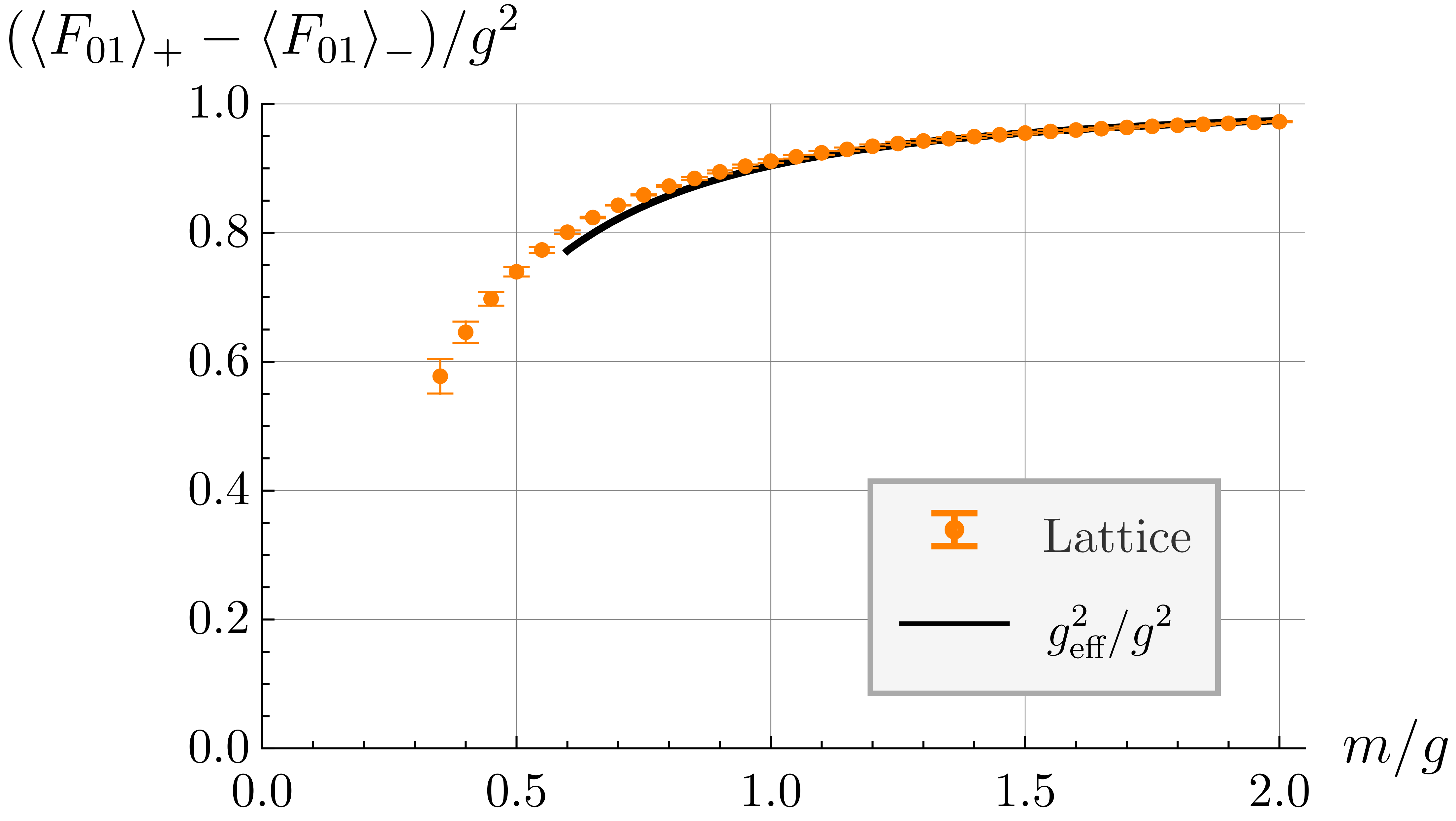
Corrections at weak coupling

- One-loop vacuum polarization: $g_{\text{eff}}^2 \simeq g^2 \left(1 - \frac{g^2}{3\pi m^2}\right)$

- Soliton mass: diagrammatic calculation. Feynman gauge warrants rainbow graph resummation [Das et. al'2012]. We use lightcone gauge $A_- = (A_0 - A_1)/\sqrt{2} = 0$ and principal value prescription to deal with IR divergences ['t Hooft'1974].

$$M_{\text{sol}} = m\left(1 - \frac{g^2}{2\pi m^2} + \frac{(3\pi^2 - 4)g^4}{96\pi^2 m^4} + \mathcal{O}\left(g^6/m^6\right)\right)$$





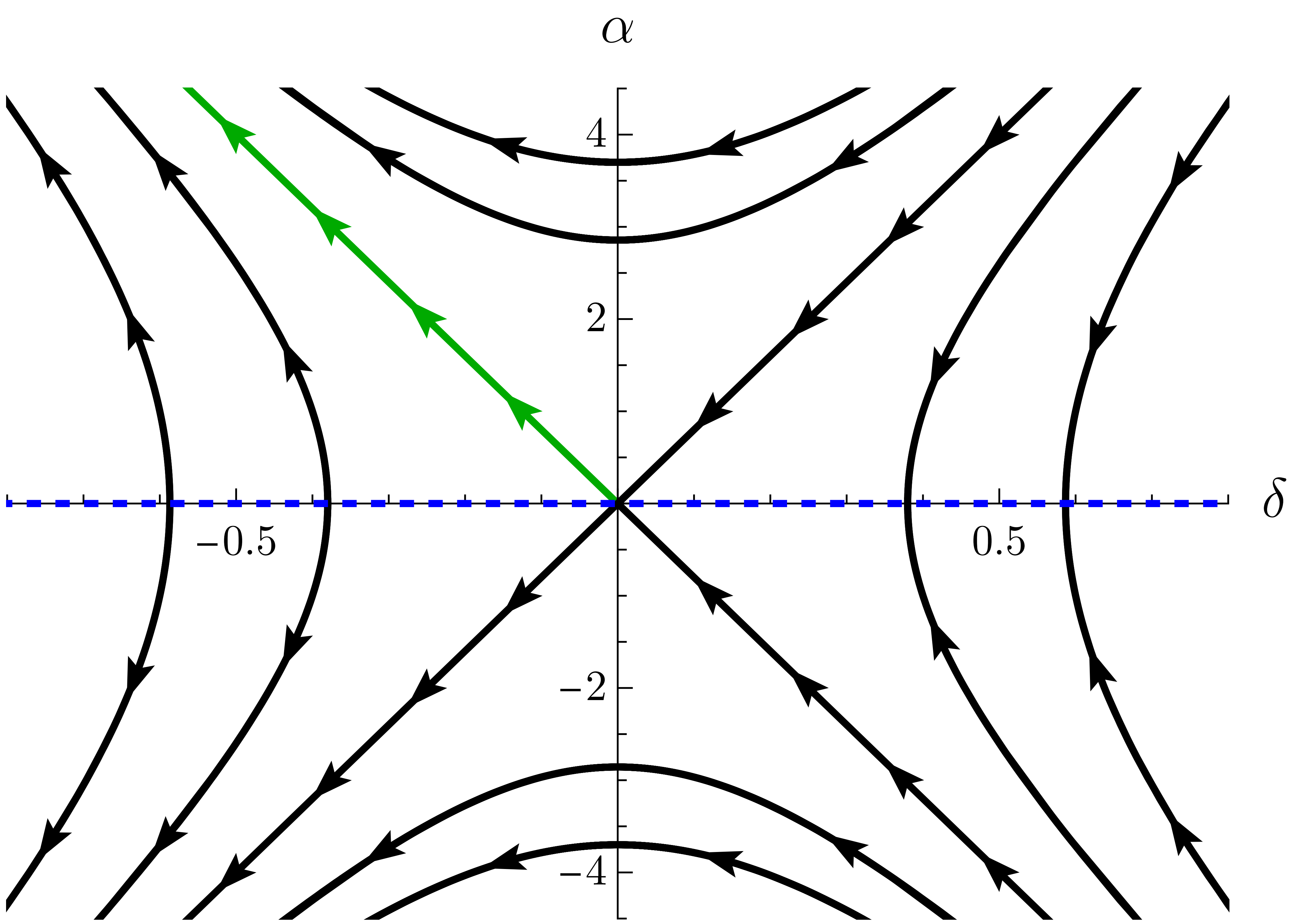
Strong coupling I

- Strong coupling: $\mathscr{L}_{\text{EFT}} \simeq \frac{1-\delta}{2}\partial_\mu\phi_-\partial^\mu\phi_- + \frac{e^{2\gamma_E}}{32\pi}\alpha\,\mu_-^2 N_{\mu_-}\cos(\sqrt{8\pi}\phi_-)$;

  from EFT $\alpha = -8\delta,\ \delta = -\frac{e^{\gamma_E}m^2}{2\pi\mu^2}I_s(\pi);\ I_s(\pi) \approx 10.1.$

  EFT gives the coupling at scale $\sim g$

- BKT RG flow [Amit et. al'1980]: $\begin{cases}\beta_\alpha = 2\alpha\delta;\\ \beta_\delta = \alpha^2/32.\end{cases}$



- Two loop result for SO(3)-invariant line
  $\alpha = -8\delta + \mathcal{O}(\delta^2)$: $\quad \beta_\delta = 2\delta^2 + 2\delta^3$;
  The theory is asymptotically free

Strong coupling II

- Dimensional transmutation: coupling blows up at $\Lambda_{\text{IR}} \simeq \frac{m}{\sqrt{\pi A_s}} e^{-A_s \frac{g^2}{m^2}}$, $A_s \approx 0.111$;

- Dimensionful quantities scale with Λ_{IR}: $M \sim \Lambda_{\text{IR}}$

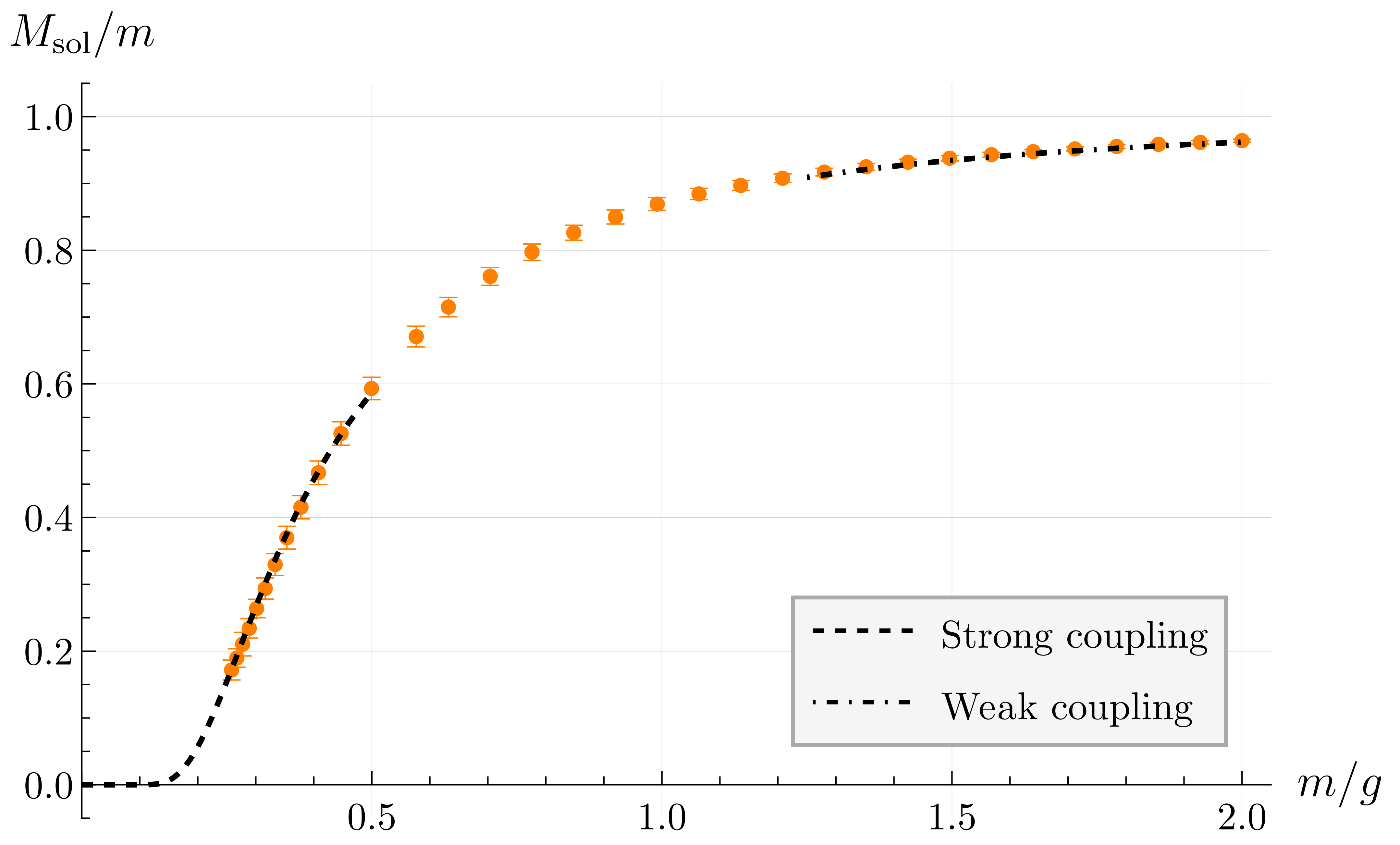
- Can be mapped to integrable SU(2) Thirring model ($\simeq$ 2-flavor Gross-Neveu) [Forgacs et. al'1980]:

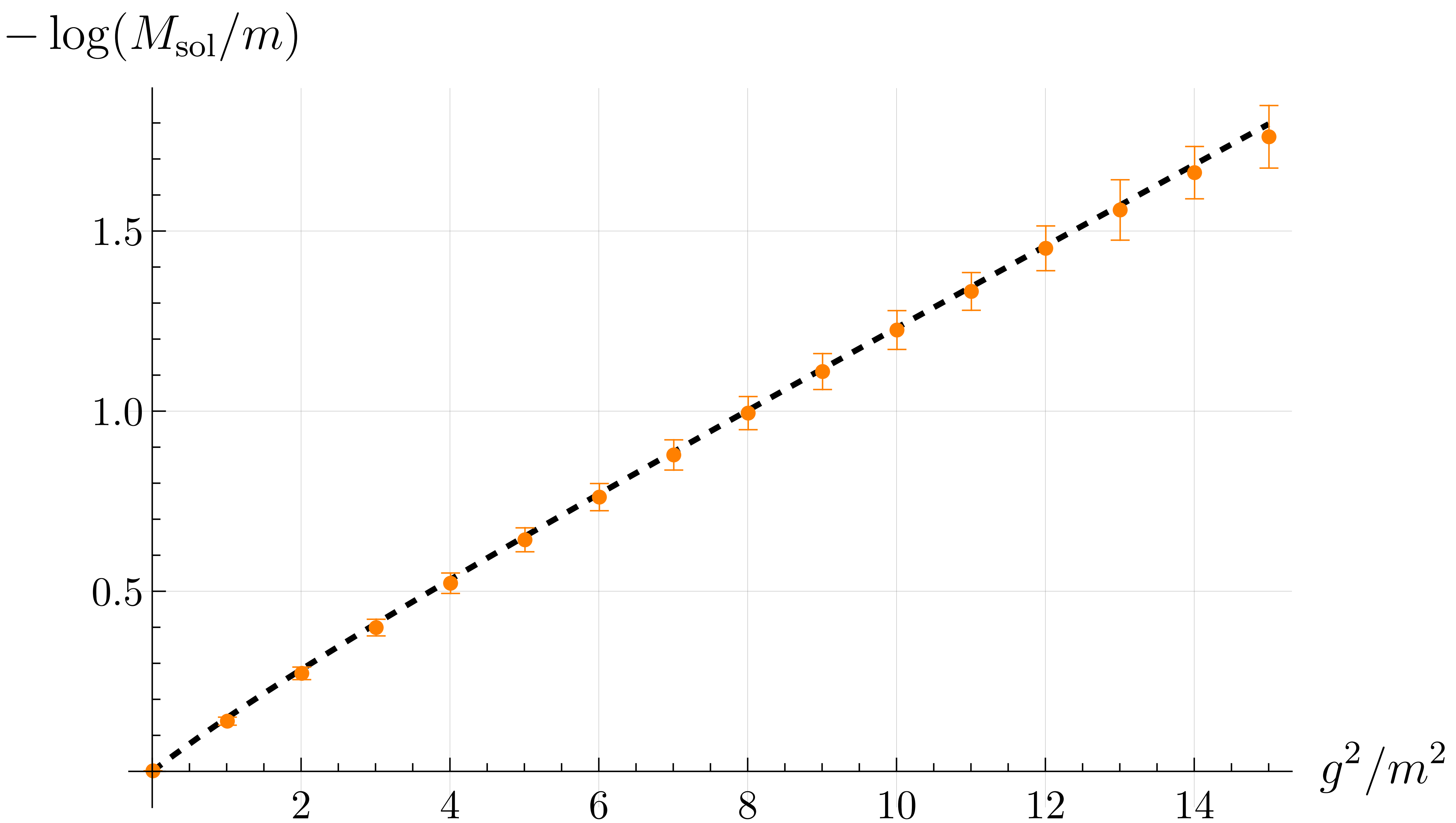
$$\mathscr{L} = i\bar{\psi}^{\alpha}\hat{\partial}\psi_{\alpha} - \lambda \left(J^{\mu} J_{\mu} + j_A^{\mu} j_{A\mu} \right); \quad J^{\mu} = \frac{1}{2}\bar{\psi}^{\alpha}\gamma^{\mu}\psi_{\alpha}; \quad j_A^{\mu} = \frac{1}{2}\bar{\psi}^{\alpha}\gamma^{\mu}(\sigma_A)_{\alpha}{}^{\beta}\psi_{\beta}$$

- Two-loop matching: express $\lambda(\mu)$ in terms of m by computing free energy at Q_3 chemical potential $m \ll h \ll \mu$. Gives $M \simeq mKe^{-A_s \frac{g^2}{m^2}}$, $K = 1.0(1)$;

Synthesis

- Weak coupling:
$$-\log\left(M_{\text{sol}}/m\right) = \frac{g^2}{2\pi m^2} - \frac{(3\pi^2-16)g^4}{96\pi^2 m^4} + \mathcal{O}\left(g^6/m^6\right)$$
- Strong coupling:
$$-\log\left(M_{\text{sol}}/m\right) = A_s\frac{g^2}{m^2} - \log K + \mathcal{O}\left(m^2/g^2\right))$$
- Best fit: $K \approx 0.91$
- [2,1] Padé approximant matching the strong coupling slope
$$-\log\left(M_{\text{sol}}/m\right)\Big|_{\text{Padé}} = \frac{g^2}{m^2}\frac{(16-3\pi^2)A_s\frac{g^2}{m^2}+48\pi A_s - 24}{(16-3\pi^2)\frac{g^2}{m^2}+96\pi^2 A_s - 48\pi};$$
$K_{\text{Padé}} = 0.853$





Discussion and future directions

- Solution to the 3rd puzzle: the charge continuously interpolates from 1 at $m \to \infty$ to 0 at $m \to 0$

- The theory is always gapped and deconfined (free charges exist) at $\theta = \pi$, $m \neq 0$; gauge-invariant vacuum excitations are in the multi-soliton continuum

- Dimensional transmutation occurs and exponentially-small mass gap is generated; the spectrum remains qualitatively the same unlike QCD or BCS

- Would be interesting to study the perturbative expansion for M_{sol} using resurgence methods

- Multiflavor Schwinger model (some WIP)