# Lattice Fermion Anomalies & CPT

arXiv:2508.17115 w/ Seiberg&Shao
arXiv:2601.01191 w/ Seiberg

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# Goals of the talk

- A systematic approach to probe lattice anomalies and match to the continuum using group theory.

- An example: order 8 reflection/time-reversal anomalies on the lattice model (2+1d staggered fermion).

- Demonstrate how lattice CPT operator helps to match the lattice anomaly and continuum anomaly.

# Anomaly

- Anomaly of symmetries, which reflects the obstruction to gauging, helps to match the UV theory and IR theory.
- It imposes constraints on low-lying states: there is no unique gapped ground state.
- These gives us a probe of the anomalies of symmetry $G$.

find a symmetric Hamiltonian $H$ with a trivially gapped ground state

→*No*

arbitrary space +
arbitrary classical gauge field of $G$:

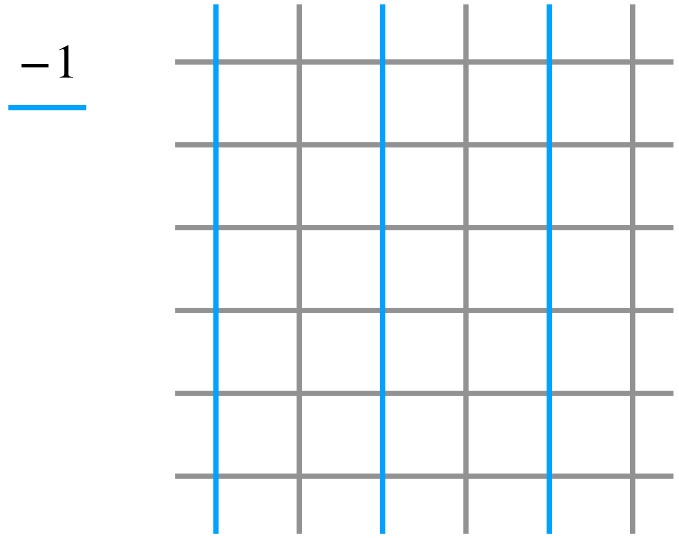
find the projective symmetry algebra

→*Yes*

- Order: the smallest # of copies s.t. anomaly-free.

# Example: 2+1d Staggered Fermion

- One Majorana fermion per site, $\mathbb{Z}_2$ background gauge field with flux $-1$ on each plaquette [Kogut, Susskind; Banks, Susskind, Kogut].



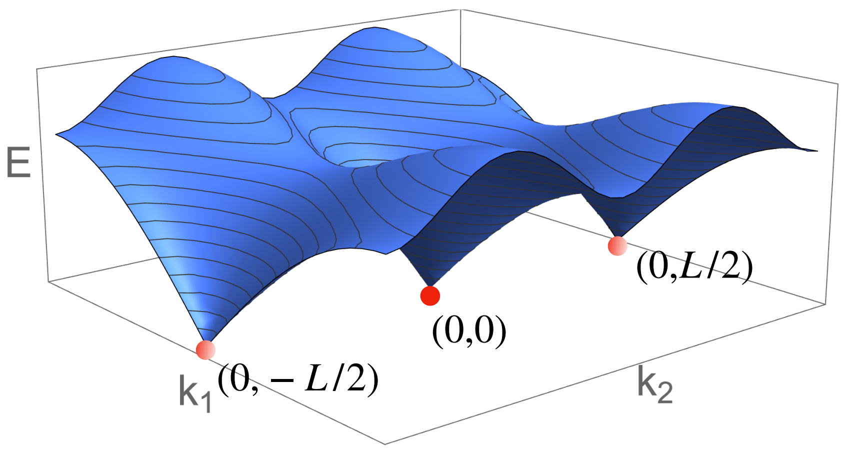
$$H = i \sum_{\vec{\ell},\mu=1,2} \eta_\mu(\vec{\ell})\psi_{\vec{\ell}+\hat{\mu}}\psi_{\vec{\ell}}$$

$$\eta_\mu(\vec{\ell}) = (-1)^{\sum_{\nu<\mu}\ell^\nu} = \begin{cases} 1 & \mu = 1 \\ (-1)^{\ell_1} & \mu = 2 \end{cases}$$

- Symmetries:

$$\mathscr{C}^4 = 1, \quad \mathscr{C}T_1\mathscr{C}^{-1} = T_2, \quad \mathscr{C}T_2\mathscr{C}^{-1} = (-1)^F T_1^{-1}, \quad T_1T_2 = (-1)^F T_2T_1,$$
$$\mathscr{R}^2 = 1, \quad T_1\mathscr{R} = (-1)^F\mathscr{R}T_1^{-1}, \quad \mathscr{R}\mathscr{C}\mathscr{R}^{-1} = \mathscr{C}^{-1},$$
$$\mathscr{T}^2 = 1, \quad \mathscr{T}T_i = (-1)^F T_i\mathscr{T}.$$

# 2+1d Staggered Fermion



- The continuum theory has two Majorana fermions in 2+1d, say $\Psi^{1,2}$.
- The internal symmetry is $\mathrm{O}(2) = U(1)^{\mathsf{J}} \rtimes \mathbb{Z}_2^{\Gamma}$.
- The spacetime symmetry is $\mathrm{Pin}_+(2{,}1)$. We focus on the subgroup $\mathrm{Pin}_+(2) \times \mathbb{Z}_2^{\Xi}$ where $\Xi = \mathsf{T} e^{i\pi \mathsf{L}}$.

The lattice symmetry is matched with the continuum ones

$$T_{1,2} \mapsto \Gamma_{1,2},$$
$$\mathscr{R} \mapsto \Gamma_2 \mathsf{R},$$
$$\mathscr{C} \mapsto e^{i\frac{\pi}{4}\mathsf{J}} e^{i\frac{\pi}{2}\mathsf{L}},$$
$$\Omega = \mathscr{C}^2 \mathscr{T} \mapsto \Xi\,.$$

$$\langle \Gamma_1, \Gamma_2 \rangle = D_4 \subset O(2),$$
$$\Gamma_1^2 = \Gamma_2^2 = 1, \quad \Gamma_1\Gamma_2 = (-1)^F \Gamma_2\Gamma_1$$

# Parity Anomaly

- In the continuum, parity anomalies refers to a class of mixed anomalies between orientation reversing spacetime symmetries and internal symmetries $G$ [Niemi, Semenoff; Redlich; Redlich, Alvarez-Gaume, Della Pietra, Moore; Rao, Yahalom].
- Notably, a Majorana fermion in 2+1d has parity anomalies of order 16 [Witten].
- In the continuum, there are various ways to study it:
  - cobordism [Kapustin, Thorngren, Turzillo, Wang],
  - modular transformation of the partition function [Hsieh, Cho, Ryu],
  - fractional anomalous momentum on the cross-cap background [Cho, Hsieh, Morimoto, Ryu; Tachikawa, Yonekura] etc.
- These methods involve non-compact/non-flat space or operations that are hard to realize on the lattice.

# Emergent Reflection Anomaly

- Only with translation and reflection, the anomaly is at most order 2. With two copies:

$$H' = i \sum_{\vec{\ell}} \psi^1_{\vec{\ell}} \psi^2_{\vec{\ell}}$$

- The corresponding anomaly of $\mathrm{O}(2) \times \mathrm{Pin}_+(2)$ is at least order 8. Therefore this anomaly is emergent.

- To see more anomalies, let's examine the rest of the lattice symmetries.

# Next

With all the lattice symmetries, we need to find

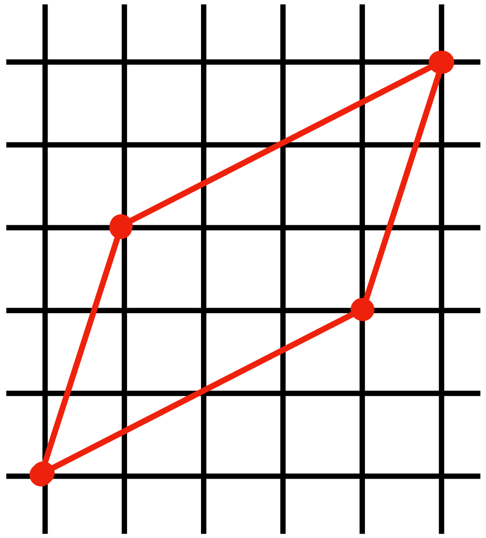
- as many as possible lattice backgrounds
- the projective phases of the unbroken symmetries
- The map from the lattice backgrounds, symmetries and projective phases to those in the continuum

Doing this case by case can be tedious and one can miss important unbroken symmetries and cases.

We need a systematic approach.

# General Framework

Start with infinite lattice $\mathbb{Z}^2$, local degrees of freedom $\psi_{\vec{\ell}}$ and Hamiltonian $H$. The symmetry is group $G$.



$$g_1 = (T_1)^4(T_2)^2$$
$$g_2 = (T_1)^1(T_2)^3$$

Then make identification to obtain a compact, flat and homogeneous manifold (2d: torus or Klein bottle).

Hamiltonian needed to be locally homogeneous. Sites identified through unitary symmetry operators, $g_I \psi_{\vec{\ell}}\, g_I^{-1} = \psi_{\vec{\ell}}$, $I = 1,2$.

The group $\mathscr{G} = \langle g_1, g_2 \rangle$ must be the fundamental group of the resulting manifold. And it gives the full list of boundary conditions.

# General Framework

What is the unbroken symmetry? It should preserve the boundary conditions.

$$g h \psi_{\vec{\ell}} h^{-1} g^{-1} = h \psi_{\vec{\ell}} h^{-1} \qquad \forall g \in \mathscr{G}$$

Therefore, all these $h$ elements form the normalizer of $\mathscr{G}$,

$$N_G(\mathscr{G}) = \left\{ h \in G \mid h g h^{-1} \in \mathscr{G}, \forall g \in \mathscr{G} \right\} .$$

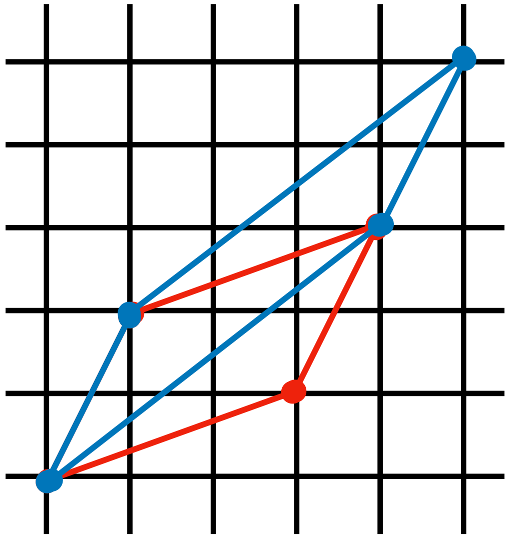
Moreover $\forall g \in \mathscr{G}$ acts trivially on $\psi_{\vec{\ell}}$. Thus we need to take a quotient to obtain the faithful symmetry,

$$G_{\text{compact}} = \frac{N_G(\mathscr{G})}{\mathscr{G}}$$

When restricting to the internal symmetries (no change to the coordinate), it becomes the centralizer,

$$G_{\text{int,comp}} = C_{G_{\text{int}}}(\mathscr{G})$$

# General Framework: Equivalence



Boundary condition $\mathscr{G} = \langle g_I \rangle \cong \pi_1(M)$.

Unbroken symmetry $G_{\text{comp}} = \dfrac{N_G(\mathscr{G})}{\mathscr{G}}$.

- **Conjugation:** when $g \notin N_G(\mathscr{G})$, $\mathscr{G} \neq g\mathscr{G}g^{-1} \equiv \mathscr{G}'$. This gives an equivalent model $\mathscr{G}'$.
  - Even though the unbroken symmetry operators might look different $N_G(\mathscr{G}) \neq N_G(\mathscr{G}')$, they are isomorphic.
- **Cycle Redefinition:** for the same boundary conditions $\mathscr{G}$ one can pick up different set of generators. They are connected by $\mathrm{Aut}(\mathscr{G})$.
  - For example, $\mathrm{Aut}(\pi_1(\mathbb{T}^2)) = \mathrm{GL}(2,\mathbb{Z})$.

# General Framework: Continuum

- We start with continuum field theory $\Psi_\alpha$ on $\mathbb{R}^2$, with symmetry $K$.
- We pick up a subgroup $\mathscr{K} = \langle k_1, k_2 \rangle$ to be the boundary conditions.
- To obtain a flat compact manifold, $\mathscr{K}$ should be the fundamental group of the resulting manifold.
- The symmetry of the theory on the compact manifold is $\dfrac{N_K(\mathscr{K})}{\mathscr{K}}$.
- Geometrically, this is (universal) group $\mathscr{K}$-covering and the symmetry becomes the isometry of the manifold.

# Questions Reformulated

1. How is the lattice symmetry $G$ matched with the continuum symmetry $K$?

2. Given the boundary conditions on the lattice $\mathscr{G}$, what is the background gauge field and topology of the manifold in the continuum $\mathscr{K}$?

3. How are the projective phases of the remaining symmetry operators on the lattice matched with those in the continuum?

# Answer by the Group Theory

- By solving the untwisted system, we match the lattice degrees of freedom $\psi$ and the continuum degrees of freedom $\Psi$.

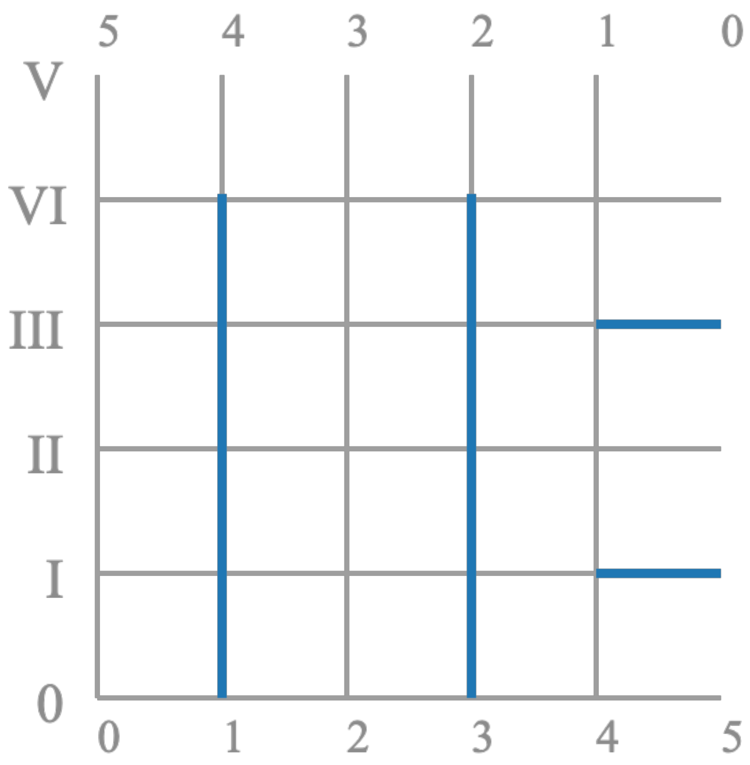
- This also gives the group homomorphism from lattice symmetry to the continuum symmetry $G \to K$.

- This homomorphism also maps the lattice BC to the continuum BC $\mathscr{G} \to \mathscr{K}$, by matching $g_I \psi g_I^{-1} = \psi$ to $k_I \Psi k_I^{-1} = \Psi$.

- The the unbroken symmetries are determined and matched.

- The projective phases of these symmetries can be analyzed on the zero modes and hence easy to match as well.

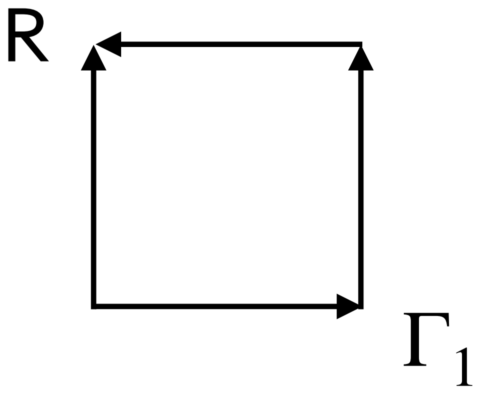
# Example: Order 8 (# copies $N_f$ even)



- The boundary condition is $\mathscr{G} = \langle T_1^5, T_2^5 \mathscr{R} \rangle$.
- Though it is not manifest, the symmetries are generated by $T_2^2, T_2\mathscr{R}, (-1)^F, \Omega$. The projective phases involve

$$\Omega(-1)^F = (-1)^{\frac{N_f}{2}}(-1)^F\Omega$$

$$\Omega^2 = (-1)^{\frac{N_f(N_f-2)}{8}}$$



- The boundary condition is $\mathscr{K} = \langle \Gamma_1, \mathrm{R} \rangle$.
- Unbroken symmetries are $\Gamma, (-1)^F, \mathrm{R}, \Xi$ and translation along direction 2. The phases are

$$\Xi(-1)^F = (-1)^{\frac{N_f}{2}}(-1)^F\Xi$$

$$\Xi^2 = (-1)^{\frac{N_f(N_f-2)}{8}}$$

# Relation to lattice CRT

$\Omega = \mathscr{R}_1\mathscr{R}_2\mathscr{T}$

$\mathscr{R}_1$, translation, $(-1)^F$

order 2

$\mathscr{R}_1$, $T_2\mathscr{R}_2\mathscr{T}$, translation, $(-1)^F$

order 8

- The lattice anti-unitary operator $T_2\mathscr{R}_2\mathscr{T}$ reflects both space and time, flows to CRT in the continuum.
- It is part of the spacetime $\pi$-rotation and itself can't have an anomaly.
- The lattice counter part $T_2\mathscr{R}_2\mathscr{T}$ probes the emergent reflection anomaly on the lattice.

# Summary

- We develop a general formalism to study homogeneous Hamiltonian lattice models on nontrivial, compact, flat spaces.
- We demonstrate the probe of an order 8 lattice anomaly of reflection/time-reversal in 2+1d staggered fermion, where in the continuum it becomes the parity anomaly in 2+1d.
- The map between the lattice and continuum anomalies is related to the CRT symmetry.
  - Many other examples in 1+1d [Seiberg, Shao, WZ].

Thank You!

# Backup Slides

# Inequivalent models

| Lattice | | | Cont | Order |
|---|---|---|---|---|
| **Boundary Condition** | ***$L_1$*** | ***$L_2$*** | **Twist** | |
| $(\ell^1, \ell^2) \sim (\ell^1 + L_1, \ell^2) \sim (\ell^1, \ell^2 + L_2)$ | even | even | $(1, 1)$ | 2 |
| | even | odd | $(1, \Gamma)$ | 4 |
| $(\ell^1, \ell^2) \sim (\ell^1 + L_1, \ell^2) \sim (-\ell^1, \ell^2 + L_2)$ | even | odd | $(1, R)$ | 4 |
| | odd | odd | $(\Gamma, R)$ | 8 |
| | even | even | $(1, \Gamma R)$ | 4 |
| $(\ell^1, \ell^2) \sim (\ell^1 + L_1, \ell^2) \sim (-\ell^1 + 1, \ell^2 + L_2)$ | even | odd | | |
| $(\ell^1, \ell^2) \sim (\ell^1 + L_1, \ell^2 + L_1)$<br>$\sim (-\ell^2 + L_2, -\ell^1 - L_2)$ | even | even | | |