Self-dual Higgs transition: Toric code and beyond

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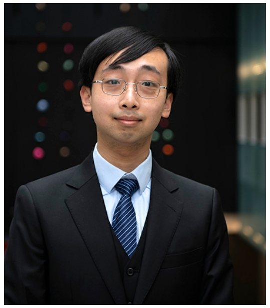
Simons Collaboration on Confinement and QCD Strings Workshop, MIT
May 19, 2026



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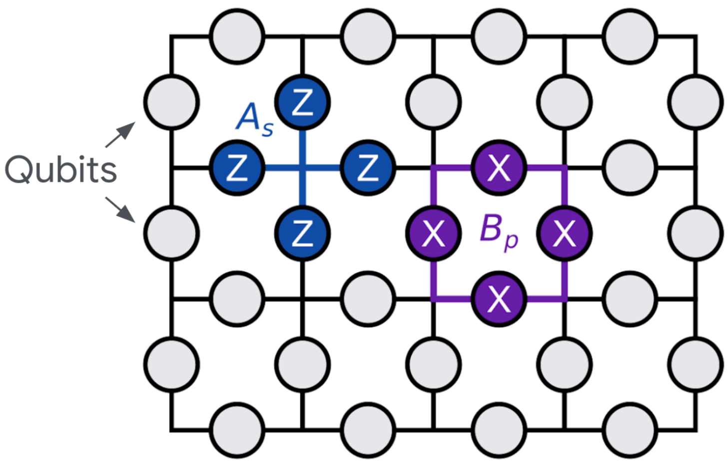
Perimeter

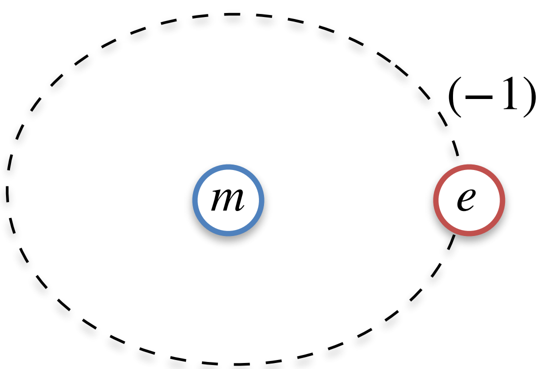
Zheng Zhou
Perimeter → Harvard

arXiv:2601.20945

$\mathbb{Z}_2$ gauge theory/$\mathbb{Z}_2$ spin liquid/Toric Code

$$H = -\sum_v XXXX - \sum_p ZZZZ$$



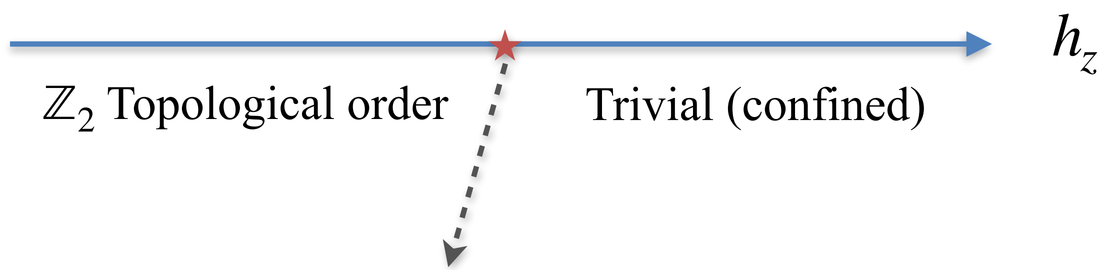


- Gapped excitations: $\{e, m, \epsilon = em\}$
- Mutual π-braiding between e, m
- Fermion statistics of ϵ
- Self-duality $\mathbb{Z}_2^{SD}$ symmetry: $e \leftrightarrow m$

Wagner; Kogut; Fradkin, Shenker; Read, Sachdev; Wen; Kitaev ……

Higgs transition of Toric Code

$$H = -\sum_v XXXX - \sum_p ZZZZ - \sum_e h_z Z_e$$



Ising* transition: $(2+1)d$ Ising CFT coupled to $\mathbb{Z}_2$ gauge field

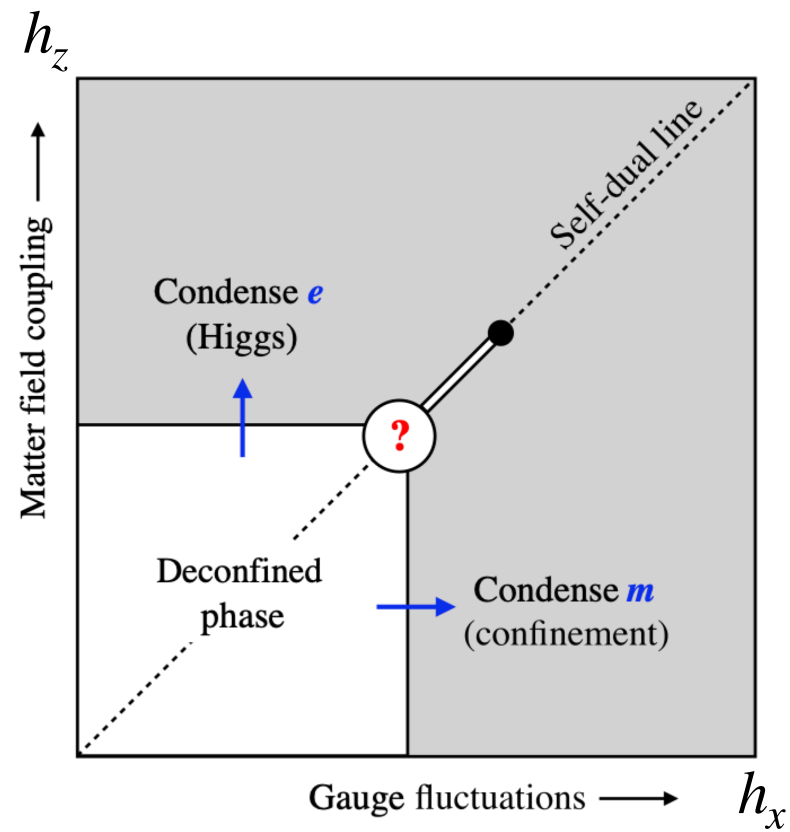
Physically: gap of e closes $\Rightarrow$ $\mathbb{Z}_2$ Higgs transition ("e condensed")

Replacing $h_z Z \to h_x X$: $e \leftrightarrow m$, still an Ising* transition ("m condensed")

Higgs with self-duality symmetry

$$H = -\sum_v XXXX - \sum_p ZZZZ - \sum_e (h_x X + h_z Z)$$

- Numerical evidence along self-dual line ($h_x = h_z$):



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$$H = -\sum_v XXXX - \sum_p ZZZZ - \sum_e (h_x X + h_z Z)$$

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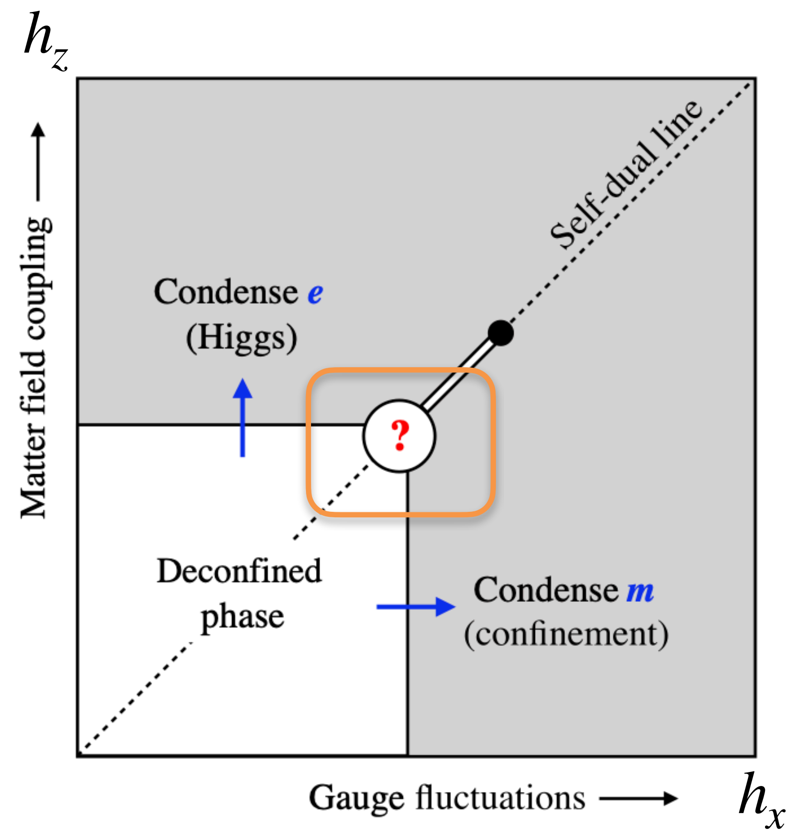
Direct, continuous transition from $\mathbb{Z}_2$ topological order to a non-topological state with spontaneously broken $\mathbb{Z}_2^{SD}$ self-duality symmetry



Tupitsyn, Kitaev, Prokof'ev, Stamp 0804.3175
Vidal, Dusuel, Schmidt 0807.0487
Wu, Deng, Prokof'ev 1201.6409
Somoza, Serna, Nahum 2012.15845
Oppenheim, Koch-Janusz, Gazit, Ringel 2311.17994

What field theory describes this?

- *Challenge: find a continuum field theory with correct phase diagram*

Direct, continuous transition from $\mathbb{Z}_2$ topological order to a non-topological state with spontaneously broken $\mathbb{Z}_2^{SD}$ self-duality symmetry

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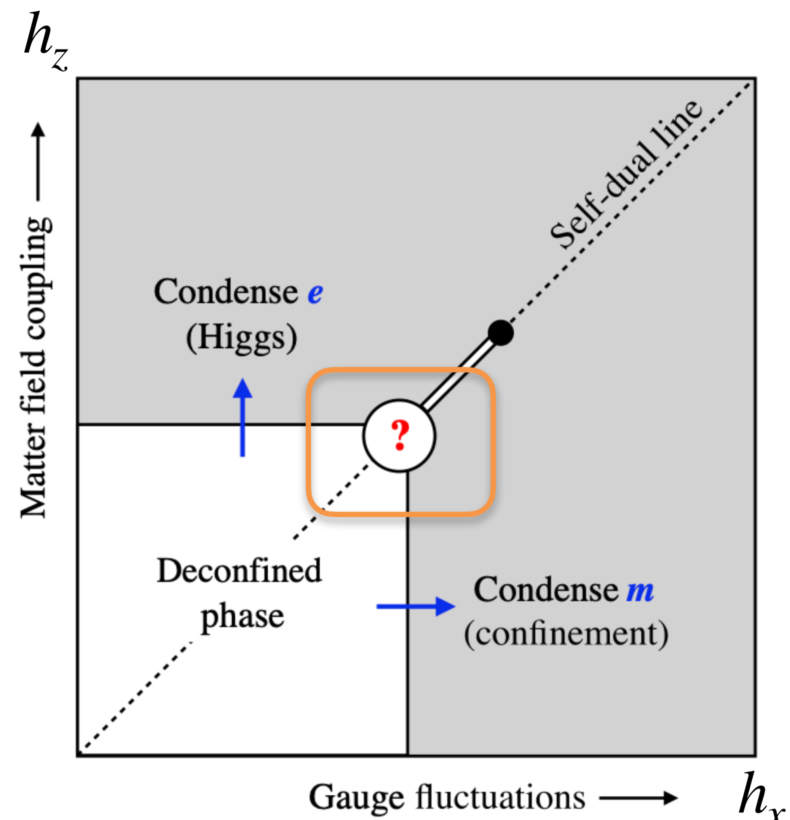


Trick: "gauging & guessing"

- Work on self-dual line, consider gauging $\mathbb{Z}_2^{SD}$

- $\mathbb{Z}_2$ topological order $\xrightarrow{\text{Gauging } \mathbb{Z}_2^{SD}}$ Double-Ising (Teo, Hughes, Fradkin, 15)

  $\{1, e, m, \epsilon = em\}$ $\qquad$ $\{1, \sigma, \psi\} \times \{1, \bar{\sigma}, \bar{\psi}\}$

- $\mathbb{Z}_2^{SD}$ spontaneously broken $\xrightarrow{\text{Gauging } \mathbb{Z}_2^{SD}}$ Trivial vacuum

After gauging $\mathbb{Z}_2^{SD}$:
Transition from double-Ising to trivial phase

Trick: “gauging & guessing”

Want: Transition from double-Ising to trivial phase

- This theory works:

$$(D_{\tilde{a}}\Phi)^2 + r\Phi^2 + \lambda\Phi^4 + \frac{1}{2g^2}\mathrm{tr}\tilde{f}_{\mu\nu}{}^2 + i\mathrm{CS}[\tilde{a}]_{2,-2}$$

- a: dynamical $SU(2)_L \times SU(2)_R$ gauge field

- Φ: scalar in $SU(2)_L \times SU(2)_R$ bi-fundamental rep

- $CS[a]_{2,-2}$: Chern-Simons term, level $+2$ (-2) for $SU(2)_L$ ($SU(2)_R$)

Matching phase diagram

$$(D_{\tilde{a}}\Phi)^2 + r\Phi^2 + \lambda\Phi^4 + \frac{1}{2g^2}\mathrm{tr}\tilde{f}_{\mu\nu}{}^2 + i\mathrm{CS}[\tilde{a}]_{2,-2}$$

$\langle\Phi\rangle \neq 0 \Rightarrow \tilde{a}^L = \tilde{a}^R$
CS terms cancel, pure $SU(2)$ YM
$\Rightarrow$ confines to trivial vacuum

Φ gapped, IR pure CS
$SU(2)_2 \times SU(2)_{-2}$ topological order
$\approx$ double Ising*

*: a twisted version of double Ising, obtained by gauging toric code + $\mathbb{Z}_2^{SD}$ SPT

Back to toric code transition

- Can "un-gauge" $\mathbb{Z}_2^{SD}$ by gauging $\mathbb{Z}_2$ one-form symmetry (charged operator: $SU(2)_L$-fund Wilson loops)

$$(D_a\Phi)^2 + r\Phi^2 + \lambda\Phi^4 + \frac{1}{2g^2}\mathrm{tr}F_{\mu\nu}{}^2 + i\mathrm{CS}[a]_{2,-2}$$

- a: dynamical $SO(4) = \dfrac{SU(2)_L \times SU(2)_R}{\mathbb{Z}_2}$ gauge field

- Φ: real scalar in $SO(4)$ vector rep (4 component)

- $CS[a]_{2,-2}$: Chern-Simons term, level $+2$ (-2) for $SU(2)_L$ ($SU(2)_R$)

Global symmetry

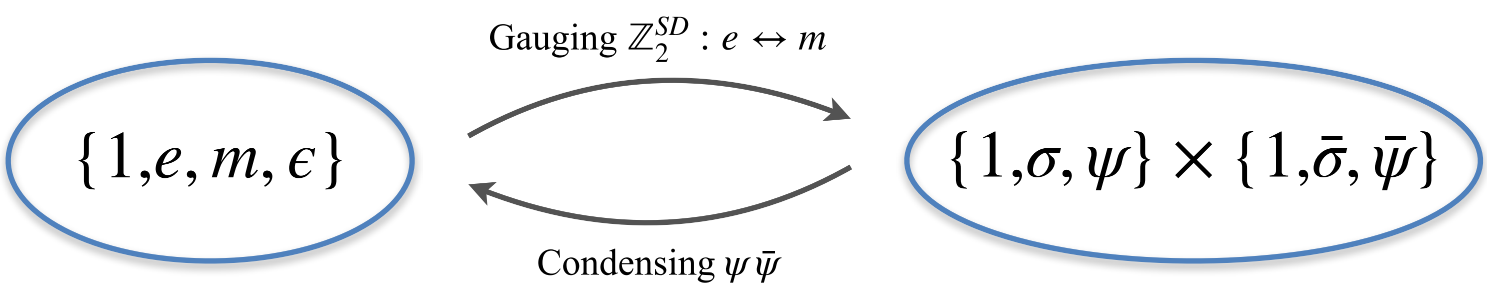
$$(D_a\Phi)^2 + r\Phi^2 + \lambda\Phi^4 + \frac{1}{2g^2}\mathrm{tr}F_{\mu\nu}{}^2 + i\mathrm{CS}[a]_{2,-2}$$

- Time-reversal $\mathbb{Z}_2^T$: exchange $SU(2)_L \leftrightarrow SU(2)_R$
- Unitary $\mathbb{Z}_2$ (to become $\mathbb{Z}_2^{SD}$): flux conservation from $\pi_1(SO(4)) = \mathbb{Z}_2$
- $\mathbb{Z}_2^{SD}$ charged local operator: $SO(4)$ monopole $\mathscr{M}_a$
- Coupling background $\mathbb{Z}_2^{SD}$ gauge field: $i\pi A^{\mathbb{Z}_2} \cup w_2^{SO(4)}$
- Sanity check: gauging $A^{\mathbb{Z}_2} \Rightarrow w_2^{SO(4)} = 0$
  $\Rightarrow a$ becomes $SU(2) \times SU(2)$ gauge field

Topological order

$$iCS[a^{SO(4)}]_{2,-2} + i\pi A^{\mathbb{Z}_2} \cup w_2^{SO(4)} + i\pi A^{\mathbb{Z}_2} \cup A^{\mathbb{Z}_2} \cup A^{\mathbb{Z}_2}$$

- $r > 0$: Φ gapped, pure $SO(4)_{2,-2}$ Chern-Simons theory
- This is just toric code in disguise
- From $SU(2)_2 \times SU(2)_{-2}$ to $SO(4)_{2,-2} = SU(2)_2 \times SU(2)_{-2}/\mathbb{Z}_2$: gauge 1-form symmetry for $\psi\bar{\psi}$ (equivalently: condensing $\psi\bar{\psi}$)



Higgs phase

- $r < 0 : \langle \Phi \rangle \neq 0$, Higgs $SO(4) \to SO(3)$ ("diagonal $SU(2)$")
- Chern-Simons terms $SU(2)_2 + SU(2)_{-2}$ cancel each other
- We are left with pure $SO(3)$ Yang-Mills

$$\frac{1}{2g^2}\mathrm{tr}{F_{\mu\nu}}^2 + i\pi A^{\mathbb{Z}_2} \cup w_2^{SO(3)}$$

SSB of $\mathbb{Z}_2^{SD}$

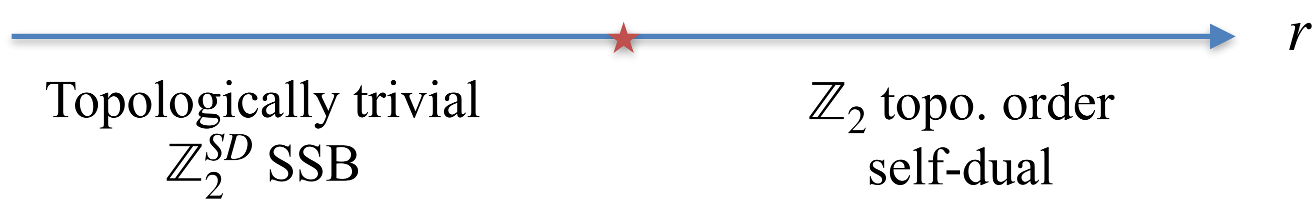
$$\frac{1}{2g^2}\mathrm{tr}F_{\mu\nu}{}^2 + i\pi A^{\mathbb{Z}_2} \cup w_2^{SO(3)}$$

- Pure $SO(3)$ Yang-Mills is expected to be gapped at IR

- $\mathbb{Z}_2$ flux symmetry will be broken:
  gauge $\mathbb{Z}_2^{SD}$ $\rightarrow w_2^{SO(3)} = 0 \rightarrow SU(2)$ pure Yang-Mills
  $\rightarrow$ should confine to a unique vacuum
  $\rightarrow$ vacuum before gauging $\mathbb{Z}_2^{SD}$ must be SSB

So, as promised:

$$(D_a\Phi)^2 + r\Phi^2 + \lambda\Phi^4 + \frac{1}{2g^2}\mathrm{tr}F_{\mu\nu}{}^2 + i\mathrm{CS}[a]_{2,-2} + i\pi A^{\mathbb{Z}_2} \cup w_2^{SO(4)}$$



Fun generalization: $SO(4)_{k,-k}$

$$(D_a\Phi)^2 + r\Phi^2 + \lambda\Phi^4 + \frac{1}{2g^2}\mathrm{tr}F_{\mu\nu}{}^2 + i\mathrm{CS}[a]_{k,-k} + i\pi A^{\mathbb{Z}_2} \cup w_2^{SO(4)}$$

Example: $k = 4$

$$(D_a\Phi)^2 + r\Phi^2 + \lambda\Phi^4 + \frac{1}{2g^2}\mathrm{tr}{F_{\mu\nu}}^2 + i\mathrm{CS}[a]_{4,-4} + i\pi A^{\mathbb{Z}_2} \cup w_2^{SO(4)}$$

Topologically trivial
$\mathbb{Z}_2^{SD}$ SSB

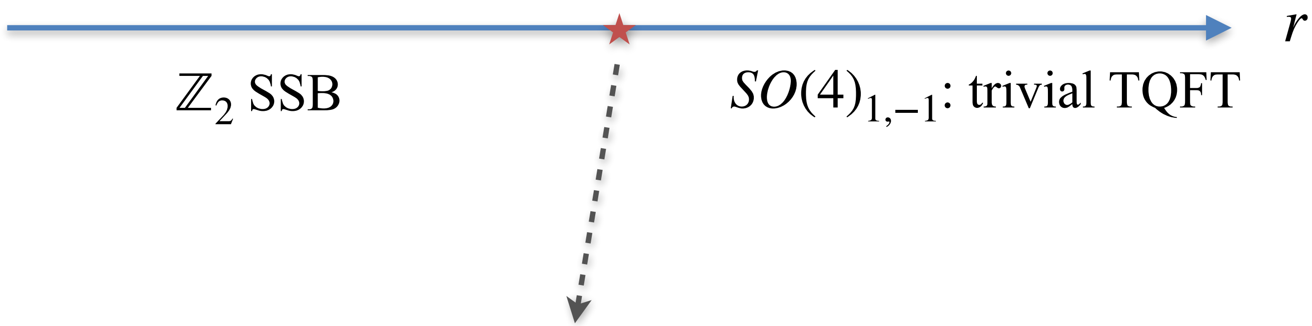
S_3 quantum double
$\mathbb{Z}_2$ exchanges non-abelian charge and flux

For large enough k, gauge field becomes classical:

$$\Delta_{\Phi^2} \to \Delta_{\epsilon}^{O(4)\mathrm{Wilson-Fisher}} \approx 1.66$$

Example: $k = 1$

$$(D_a\Phi)^2 + r\Phi^2 + \lambda\Phi^4 + \frac{1}{2g^2}\mathrm{tr}F_{\mu\nu}{}^2 + i\mathrm{CS}[a]_{1,-1} + i\pi A^{\mathbb{Z}_2} \cup w_2^{SO(4)}$$



Natural conjecture: **an IR dual of $3d$ Ising**!

Ising spin $\phi \leftrightarrow \mathscr{M}_{SO(4)}$

Similar to particle-vortex duality for complex scalar

Summary

$$(D_a\Phi)^2 + r\Phi^2 + \lambda\Phi^4 + \frac{1}{2g^2}\mathrm{tr}F_{\mu\nu}{}^2 + i\mathrm{CS}[a]_{2,-2} + i\pi A^{\mathbb{Z}_2} \cup w_2^{SO(4)}$$

r

Trivial
$\mathbb{Z}_2^{SD}$ SSB

$\mathbb{Z}_2$ topo. order
self-dual

To do:

1. Monopole

   In progress with Matthew Blakenem, Yunchao Zhang

2. Generalization, e.g. $D(S_3)$

   In progress with Dachuan Lu, Ashvin Vishwanath

3. Defects

   In progress with Ryan Lanzetta

Zheng Zhou
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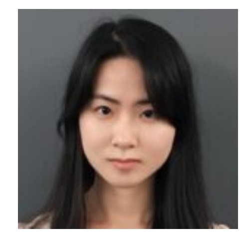

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Thank you!