Intro to TMDs Lecture 2

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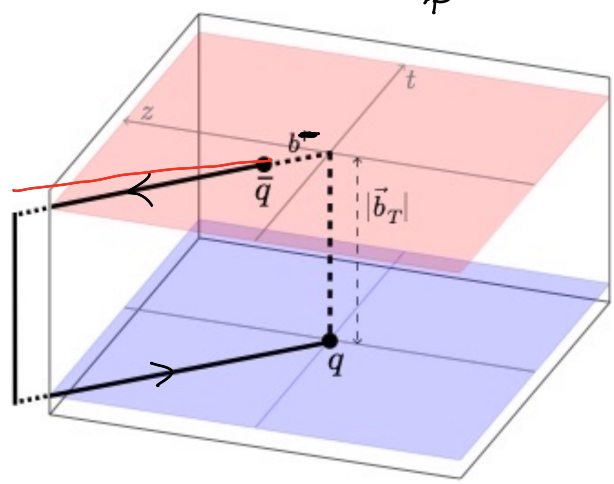
Review

$$\tilde f_{i/p}(x, b_T, \mu, y) = \lim_{\substack{\epsilon\to 0\\ \tau\to 0}} \underbrace{Z^i_{uv}(\mu, y, \epsilon)}_{\text{UV counterterm}} \; \underbrace{\frac{\tilde f^{(u)}_{i/p}(x, b_T, \epsilon, \tau, xP^+)}{\tilde S^{subt}(b_T, \epsilon, \tau)}}_{\text{bare } \tilde B(x, b_T, \epsilon, \tau, xP^+)} \sqrt{\tilde S(b_T, \epsilon, \tau)}$$

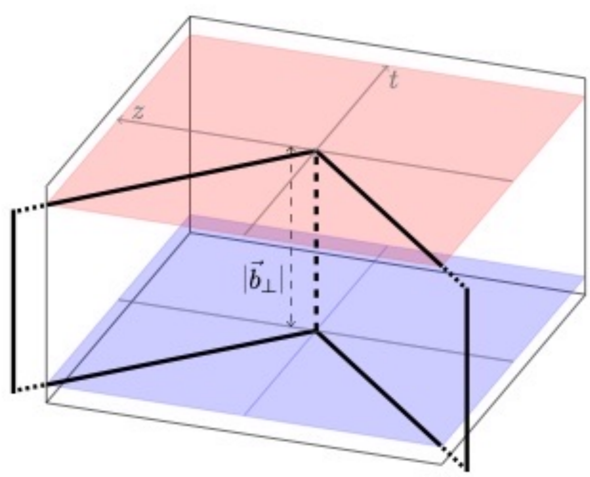
$$\tilde f^{(u)}_{i/p} = \int \frac{db^-}{2\pi} e^{-ib^- x P^+} \langle p | \left[\bar\psi_i(b^\mu) \frac{\gamma^+}{2} W_{\sqsupset} \psi_i(0) \right]_\tau | p \rangle$$

$W_{\sqsupset}$



$$\tilde S(b_T, \epsilon, \tau) = \frac{1}{N_c} \langle 0 | \left[\mathrm{Tr}\; W_{\triangleright}(b_T) \right]_\tau | 0 \rangle$$

$W_{\triangleright}$



Layer (4) Rapidity Regulators

Why do we need τ ?

regulators

$$\int_{q_T}^{Q} \frac{dk^+}{k^+} = \lim_{\tau\to 0}\Big[\underbrace{\int_{0}^{Q} \frac{dk^+}{k^+} R_c(k,\tau)}_{\text{collinear approx}} + \underbrace{\int_{q_T}^{\infty} \frac{dk^+}{k^+} R_s(k,\tau)}_{\text{soft approx}} \Big]$$

must expand to derive factorization

Examples

- Collins, Space-Like Wilson Lines $\quad \frac{\sqrt{\tilde S}}{\tilde S^{subt}} \to \frac{1}{\sqrt{\tilde S}}$

light-cone $\overset{+\ \ -\ \ \perp}{(0, 1, 0_T)} \to (-e^{2y_B}, 1, 0_T)$ with $y_B \to -\infty$

$$\tilde f_{i/p}(x, b_T, \mu, \zeta) = \lim_{\epsilon\to 0} Z_{uv}(\mu, \zeta, \epsilon) \lim_{y_B\to-\infty} \frac{\tilde f^{(u)}_{i/p}(x, b_T, \epsilon, y_B, xP^+)}{\sqrt{\tilde S(b_T, \epsilon, 2y_n - 2y_B)}}$$

here $\zeta = 2(xP^+)^2 e^{-2y_n}$

- η regulator   Chiu, Jain, Neill, Rothstein $\eta \to 0$

introduce $|\sqrt{2}k^+/\nu|^{-\eta}$ in $\overset{\text{Wilson Lines}}{W_\Gamma} \to \int \frac{dk^+}{k^+}\left|\frac{\sqrt{2}k^+}{\nu}\right|^{-\eta}$

$|k^z/\nu|^{-\eta/2}$ in $W_{\gtrless} \to \int \frac{dk^+dk^-}{k^+k^-}\left|\frac{k^z}{\nu}\right|^{-\eta}$

$$\tilde f_{i/p}(x, b_T, \mu, \zeta) = \lim_{\epsilon\to 0} Z_{uv}(\mu, \zeta, \epsilon) \lim_{\eta\to 0} \tilde f^{(u)}(x, b_T, \epsilon, \eta, xP^+)\sqrt{\tilde S(b_T, \epsilon, \eta)}$$

$\tilde S^{subt} = 1$ here

- different $\tilde{f}^{(u)}$, $\tilde{S}$ but <u>same</u> $\tilde{f}_{i/p}$ & Z_{uv}
- many constructions (§2.4.1) yield same $\tilde{f}_{i/p}$, but not all (§2.5)

One-Loop Illustration of Concepts

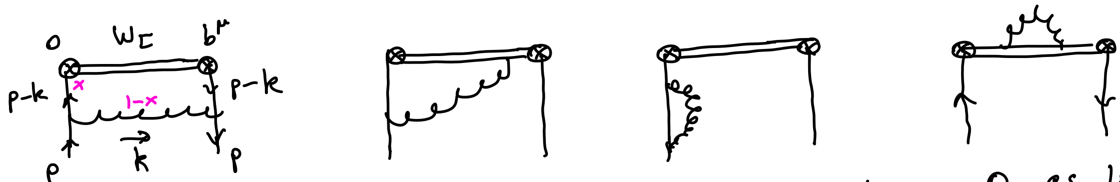
proton → quark state $\quad p^\mu = (p^+, 0, 0)$, $p^2 = 0$

$d = 4 - 2\epsilon$ for UV & IR , Feynman Gauge

η - regulator: $\tilde{f}^{(u)}$, $\tilde{S}$, $\tilde{S}^{subt} = 1$

bare

$$\tilde{f}^{(u)}_{q/q'}(x, b_T, \epsilon, \eta, xp^+) = \int \frac{db^-}{2\pi}\, e^{-ib^- x p^+} \langle q'(p) | \left[\bar{\psi}(b^\mu)\, W_\sqsubset\, \frac{\gamma^+}{2}\, \psi(0) \right]_\eta | q'(p) \rangle$$



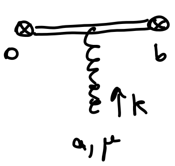
+ mirror

(a) (b)

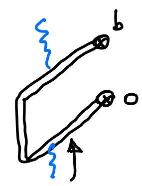
0, $\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}}$ (careful)

0 as light-like lines $n_b^\mu n_{b\mu} = 0$

transv @ ∞ = 0



$$= -g_0 \frac{n_b^\mu t^a e^{-ik\cdot b}}{n_b\cdot k + i0} + \frac{g_0\, n_b^\mu t^a}{n_b \cdot k - i0}$$



$n_b = \frac{1}{\sqrt{2}}(1, 0, 0, -1)$

$$M_a = -i g_0^2 C_F \int d^dk \int db^-\, e^{-ib^- x p^+} e^{i(p-k)\cdot b} \frac{\bar{u}\, \gamma^\mu (\not{p} - \not{k}) \gamma^+ (\not{p} - \not{k}) \gamma_\mu u}{2\left[(p-k)^2 + i0\right]^2 (k^2 + i0)} \quad R_c$$

$$M_b = -2 i g_0^2 C_F \int d^dk \underbrace{\int db^-\, e^{-ib^- x p^+} e^{i(p-k)\cdot b}}_{\delta[(1-x)p^+ - k^+]\, e^{i b_T \cdot k_T}} \frac{\bar{u}\, \gamma^+ (\not{p} - \not{k}) \gamma^+ u}{2(k^+ + i0)\left[(p-k)^2 + i0\right](k^2 + i0)} \quad R_c$$

+ scaleless

$\overline{MS}$ $\quad g_0 = Z_g\, \mu^\epsilon\, g(\mu) \left(\frac{e^{\gamma_E}}{4\pi}\right)^{\epsilon/2}$

$$M_a + M_b = \frac{\alpha_s(\mu)\, C_F}{2\pi}\left[\underbrace{\frac{1+x^2}{1-x}}_{P_{qq}(x)} - \epsilon(1-x)\right]\Gamma(-\epsilon)\left(\frac{b_T^2\mu^2}{4e^{-\gamma_E}}\right)^{\epsilon} R_c$$

singular as $x \to 1$
ie $k^+ \to 0$

η-reg $\quad R_c = w^2\left|\frac{\sqrt{2}\,k^+}{\nu}\right|^{-\eta} = w^2\left(\frac{(1-x)\,p^+}{\nu/\sqrt{2}}\right)^{-\eta}$

Expand
$\eta \to 0$
$\epsilon \to 0$

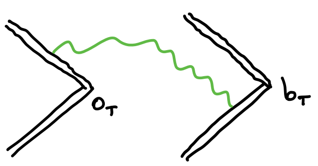
use: $(1-x)^{-1-\tau} = -\frac{1}{\tau}\,\delta(1-x) + \left(\frac{1}{1-x}\right)_+ + \mathcal{O}(\tau)$

$$(1+x^2)(1-x)^{-1-\tau} = -\left(\frac{2}{\tau}+\frac{3}{2}\right)\delta(1-x) + \left(\frac{1+x^2}{1-x}\right)_+ + \mathcal{O}(\tau)$$

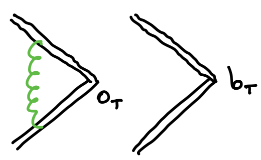
where $\quad [f(x)]_+ = f(x),\ x \neq 1$

$$\int_0^1 dx\, [f(x)]_+\, g(x) = \int_0^1 dx\, f(x)\,[g(x) - g(1)], \quad \text{any } g$$

$$\tilde{S}(b_T, \epsilon, \tau) = \frac{1}{N_c}\langle 0 | \left[\mathrm{Tr}\ W_{\geqslant}(b_T)\right]_\tau | 0\rangle$$



+ mirror



+ mirror

$$R_s^{tot} = w^2\left|\frac{k^+ - k^-}{\nu/\sqrt{2}}\right|^{-\eta}$$

$$0 = \frac{1}{\epsilon} - \frac{1}{\epsilon}$$

$\downarrow$

$$M_s = 2 g_0^2 C_F \int d^dk\ e^{i b_T\cdot k_T}\ \frac{-i}{(2k^+k^- - k_T^2 + i0)}\ \frac{2}{(k^+ - i0)(-k^- + i0)}\ R_s$$

$k^+ = \frac{k_\perp^2}{2k^-} - i0 \quad$ for $k^- > 0$

$$= \frac{g_0^2 C_F}{4\pi}\int d^{d-2}k_T\ \frac{e^{i b_T\cdot k_T}}{k_T^2}\int_0^\infty \frac{dk^-}{k^-}\ w^2\left|\frac{k_T^2}{2k^-} - k^-\right|^{-\eta}\left(\frac{\nu}{\sqrt{2}}\right)^{\eta}$$

bare

$$\tilde{f}_{q/q}^{(u)(1\text{-loop})} = \frac{\alpha_s(\mu)\,C_F}{2\pi}\Big\{-\left(\frac{1}{\epsilon_{IR}} + L_b\right)[P_{qq}(x)]_+ + (1-x)$$
$$+\ \delta(1-x)\left(\frac{1}{\epsilon_{UV}} + L_b\right)\left(\frac{2}{\tau} + \frac{3}{2} + \ln\frac{\nu^2}{\zeta}\right) + \mathcal{O}(\tau) + \mathcal{O}(\epsilon)\Big\}$$

$L_b \equiv \ln\frac{b_T^2\mu^2}{b_0^2}, \quad b_0 = 2e^{-\gamma_E}, \quad \zeta = 2(xp^+)^2$

$$\tilde{S}_q^{(1\text{-loop})} = \frac{\alpha_s(\mu)\, C_F}{2\pi}\left[\frac{2}{\epsilon_{uv}^2} + 2\left(\frac{1}{\epsilon_{uv}} + L_b\right)\left(\frac{-2}{\tau} + \ln\frac{\mu^2}{\nu^2}\right) - L_b^2 - \frac{\pi^2}{6}\right] + O(\tau) + O(\epsilon)$$

$$Z_{uv}^q = 1 - \frac{\alpha_s(\mu)\, C_F}{2\pi}\left[\frac{1}{\epsilon_{uv}^2} + \frac{1}{\epsilon_{uv}}\left(\frac{3}{2} + \ln\frac{\mu^2}{\zeta}\right)\right] \quad \text{in } \overline{MS}$$

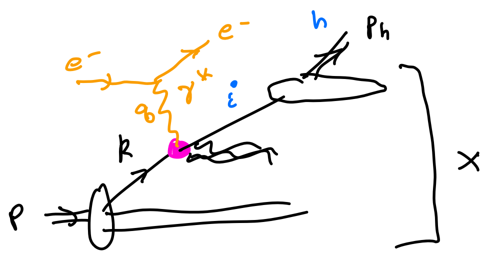
$$\tilde{f}^{(1\text{-loop})}(x, b_T, \mu, \zeta) = Z_{uv}^q\, \tilde{f}^{(u)}_{q/q}\, \sqrt{\tilde{S}}$$

$$= \frac{\alpha_s(\mu)\, C_F}{2\pi}\left[-\left(\frac{1}{\epsilon_{IR}} + L_b\right)[P_{qq}(x)]_+ + (1-x) - \frac{L_b^2}{2} + L_b\left(\frac{3}{2} + \ln\frac{\mu^2}{\zeta}\right) - \frac{\pi^2}{12}\right]$$

- expected IR div.
- $\frac{1}{\tau}$'s cancel, $\ln \nu^2$ cancel
- Z_{uv}^q has correct form $\rightarrow$ RGE (Duff's Lectures)

Semi-Inclusive DIS (SIDIS)

$e^-(\ell) + p \to e^- + h(P_h) + X$



$$q^2 = -Q^2 < 0$$

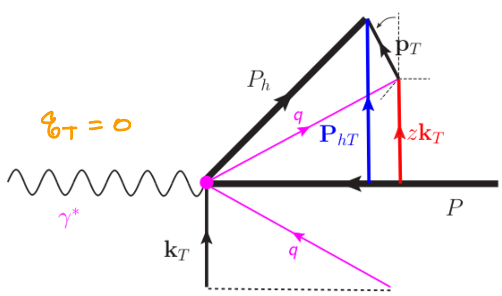
$$x = \frac{Q^2}{2P\cdot q}, \quad y = \frac{P\cdot q}{P\cdot \ell}, \quad z_h = \frac{P\cdot P_h}{P\cdot q}$$

Photon-Hadron Frame

$$P_{hT} \neq 0$$

$$P_{hT} = z_h k_T + p_T$$

$q_T = 0$



$P_T = 0$

$$\sigma \sim \int d^2k_T\, d^2p_T\, \delta^2(P_{hT} - z_h k_T - p_T)\, D_{i/h}(z_h, p_T)\, f_{i/p}(x, k_T)$$

- $D_{h/i}$ describes fragmentation of quark i to hadron h with momentum fraction z_h & trans. mom. p_T relative to quark

$$\frac{d\sigma}{dx\,dy\,dz_h\,dP_{hT}^2} = \sigma_0^{SIDIS}(x,y,Q)\, H_{ii}(Q^2,\mu) \int_0^{2\pi} d\phi_h \int d^2b_T\, e^{i b_T\cdot P_{hT}/z_h}$$

$$\times\ \tilde f_{i/p}(x,b_T,\mu,\zeta_p)\, \underbrace{\tilde D_{h/i}(z_h,b_T,\mu,\zeta_h)}_{\text{TMD frag. fn.}}$$

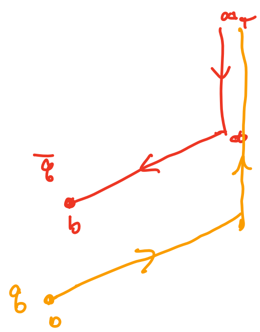
$$\tilde D_{h/i}(z,b_T,\mu,\zeta) = \lim_{\epsilon\to 0} Z^i_{uv}(\mu,\zeta,\epsilon) \lim_{\tau\to 0} \frac{\tilde D^{(u)}_{h/i}(z,b_T,\epsilon,\tau,P_h^+/z)}{\tilde S^{subt}(b_T,\epsilon,\tau)}$$

$$\times \sqrt{\tilde S(b_T,\epsilon,\tau)}$$

$$\tilde D^{(u)}_{h/i} = \frac{1}{4N_c z}\, \mathrm{tr} \int \frac{db^-}{2\pi}\, e^{ib^-(P_h^+/z)}\, \gamma^+_{\alpha\alpha'}$$

$$\times \sum_X \langle 0|\, [(W_\lrcorner\, \psi_i^\alpha)(b)]_\tau\, |h(P_h)X\rangle\langle X h(P_h)|\, [\bar\psi_i^{\alpha'} W_\urcorner)(0)]_\tau\, |0\rangle$$

$$b = (0, b^-, b_T)$$



Wilson lines go to $+\infty$

- In fact $\tilde f_{i/p}$ in SIDIS also outgoing Wilson lines $+\infty$ $W_\urcorner$ (rather than $-\infty$ $W_\llcorner$)

Spin Polarized TMDs

- consider pol. protons & pol. quarks
- Have 8 TMD PDFs at leading order $\frac{q_T}{Q} \ll 1$

Consider

$$\Phi_{\alpha\alpha'} = \int db^- d^2b_T \, e^{-ib^- x p^+} e^{i b_T \cdot k_T} \langle p(P,S) | [\bar{\psi}^i_{\alpha}(b) \, W_\Gamma \, \psi^i_{\alpha'}(0)]_\tau | p(P,S) \rangle$$

Spin Vector S^μ $\qquad S^\mu = S_L \frac{(p^- n_a^\mu - p^+ n_b^\mu)}{M} + S_T^\mu$

$$n_{a,b} = \frac{(1,0,0,\pm 1)}{\sqrt{2}}$$

$$-S^2 = S_L^2 + S_T^2 = \begin{cases} +1 & \text{pure state} \\ <1 & \text{mixed state} \end{cases}$$

$$\bar{u} \gamma^\mu u = 2 p^\mu$$
$$\bar{u} \gamma^\mu \gamma_5 u = 2 m S^\mu \quad , \quad u(p,s)\, \bar{u}(p,s) = \frac{\not{p} + M}{2} (1 + \gamma_5 \not{S})$$

unpol. $\quad$ pol.

Constraints on $\Phi_{\alpha\alpha'}$

- no S^μ or linear in S^μ
- hermiticity $\quad \Phi^\dagger = \gamma_0 \Phi \gamma_0$
- Parity $\quad \Phi^P = \gamma_0 \Phi \gamma_0$
- "Good Quark Fields" for Leading order terms : $\frac{\gamma^- \gamma^+}{2} \psi^i = \psi^i$

(see SCET)

$$\Phi = \frac{1}{2} \Big\{ f_1 \gamma^- - f_{1T}^\perp \epsilon_T^{\rho\sigma} \frac{k_{T\rho}}{m} S_{T\sigma} \gamma^- + \Big(S_L g_1 - \frac{k_T \cdot S_T}{m} g_{1T}^\perp \Big) \gamma_5 \gamma^-$$
$$+ h_1 \not{S}_T \gamma^- \gamma_5 + \Big(S_L h_{1L}^\perp - \frac{k_T \cdot S_T}{M} h_{1T}^\perp \Big) \frac{\not{k}_T \gamma^- \gamma_5}{M} + i h_1^\perp \frac{\not{k}_T \gamma^-}{M} \Big\}$$

8 TMDs

Leading Quark TMDPDFs

Nucleon Spin   Quark Spin

		Quark Polarization		
		Un-Polarized (U)	**Longitudinally Polarized (L)**	**Transversely Polarized (T)**
Nucleon Polarization	**U**	f_1 = Unpolarized		$h_1^\perp$ = − Boer-Mulders
	L		g_1 = − Helicity	$h_{1L}^\perp$ = − Worm-gear
	T	$f_{1T}^\perp$ = − Sivers	$g_{1T}^\perp$ = − Worm-gear	h_1 = − Transversity $h_{1T}^\perp$ = − Pretzelosity

Under T-reversal $S^\mu \to -S^\mu$

$\boxed{TP}$ • naively $f_{1T}^\perp$, $h_1^\perp$ <u>odd</u>, rest <u>even</u>

• but also switches Wilson lines $\underset{+\infty}{W_{\sqsupset}}^{\,SIDIS} \leftrightarrow \underset{-\infty}{W_{\sqsubset}}^{\,DY}$

$$\therefore \quad (f_{1T}^\perp)^{SIDIS} = -(f_{1T}^\perp)^{DY}$$
$$(h_1^\perp)^{SIDIS} = -(h_1^\perp)^{DY}$$

others equal

Famous SIDIS sign-flip

Brodsky, Hwang, Schmidt.
Collins
...

Contracting $f^{[\Gamma]}_{i/p_S}$, $\Gamma_{\alpha\alpha'}$ we can write $\left(\kappa = \begin{matrix} +1 & DY \\ -1 & SIDIS \end{matrix}\right)$

$|\vec{k}_T|$

$$
\begin{aligned}
f_{i/p_S}^{[\gamma^+]}(x,\mathbf{k}_T,\mu,\zeta) &= f_1(x,k_T) - \frac{\epsilon_T^{\rho\sigma}k_{T\rho}S_{T\sigma}}{M}\kappa f_{1T}^\perp(x,k_T)\,,\\
f_{i/p_S}^{[\gamma^+\gamma_5]}(x,\mathbf{k}_T,\mu,\zeta) &= S_L\, g_1(x,k_T) - \frac{k_T\cdot S_T}{M} g_{1T}^\perp(x,k_T)\,,\\
f_{i/p_S}^{[i\sigma^{\alpha+}\gamma_5]}(x,\mathbf{k}_T,\mu,\zeta) &= S_T^\alpha h_1(x,k_T) + \frac{S_L k_T^\alpha}{M} h_{1L}^\perp(x,k_T)\\
&\quad - \frac{\mathbf{k}_T^2}{M^2}\left(\frac{1}{2}g_T^{\alpha\rho} + \frac{k_T^\alpha k_T^\rho}{\mathbf{k}_T^2}\right)S_{T\rho}h_{1T}^\perp(x,k_T) - \frac{\epsilon_T^{\alpha\rho}k_{T\rho}}{M}\kappa h_1^\perp(x,k_T)
\end{aligned}
\qquad (2.124)
$$

For later use, the Fourier transform

$|b_T|$

$$\tilde f_{i/p_S}^{[\gamma^+]}(x,\mathbf{b}_T,\mu,\zeta) = \tilde f_1(x,b_T) + i\epsilon_{\rho\sigma} b_T^\rho S_T^\sigma M \tilde f_{1T}^{\perp}(x,b_T)\,,$$
$$\tilde f_{i/p_S}^{[\gamma^+\gamma_5]}(x,\mathbf{b}_T,\mu,\zeta) = S_L \tilde g_1(x,b_T) + i b_T\cdot S_T M \tilde g_{1T}^{\perp}(x,b_T)\,, \qquad (2.127)$$
$$\tilde f_{i/p_S}^{[i\sigma^{\alpha+}\gamma_5]}(x,\mathbf{b}_T,\mu,\zeta) = S_T^\alpha \tilde h_1(x,b_T) - iS_L b_T^\alpha M \tilde h_{1L}^{\perp}(x,b_T) + i\epsilon^{\alpha\rho} b_{\perp\rho} M \tilde h_1^{\perp}(x,b_T)$$
$$+ \frac{1}{2}\mathbf{b}_T^2 M^2 \left(\frac{1}{2} g_T^{\alpha\rho} + \frac{b_T^\alpha b_T^\rho}{\mathbf{b}_T^2}\right) S_{\perp\rho} \tilde h_{1T}^{\perp}(x,b_T)\,.$$

The k_T prefactors complicate the Fourier transform:

$$\tilde f_1(x,b_T) \equiv \tilde f_1^{(0)}(x,b_T)\,, \qquad \tilde f_{1T}^{\perp}(x,b_T) \equiv \tilde f_{1T}^{\perp(1)}(x,b_T)\,, \qquad \tilde h_{1T}^{\perp}(x,b_T) \equiv \tilde h_{1T}^{\perp(2)}(x,b_T)$$
$$\tilde g_{1L}(x,b_T) \equiv \tilde g_{1L}^{(0)}(x,b_T)\,, \qquad \tilde h_1^{\perp}(x,b_T) \equiv \tilde h_1^{\perp(1)}(x,b_T)\,,$$
$$\tilde h_1(x,b_T) \equiv \tilde h_1^{(0)}(x,b_T) \qquad \tilde g_{1T}(x,b_T) \equiv \tilde g_{1T}^{(1)}(x,b_T)\,, \qquad (2.128)$$
$$\tilde h_{1L}^{\perp}(x,b_T) \equiv \tilde h_{1L}^{\perp(1)}(x,b_T)\,.$$

$$\tilde f^{(n)}(x,b_T,\mu,\zeta) \equiv n!\left(\frac{-1}{M^2 b_T}\partial_{b_T}\right)^n \tilde f(x,b_T,\mu,\zeta) \qquad (2.129)$$
$$= \frac{2\pi\, n!}{(b_T M)^n}\int_0^\infty \mathrm{d}k_T\, k_T \left(\frac{k_T}{M}\right)^n J_n(b_T k_T)\, f(x,k_T,\mu,\zeta)$$

Bessel Fn. order n

A similar spin decomposition can be done for

- Quark TMD FFs
- Gluon TMD PDFs & FFs

see § 2.7 of Handbook

We'll need:

Leading Quark TMDFFs — Hadron Spin — Quark Spin

	Quark Polarization		
	Un-Polarized (U)	Longitudinally Polarized (L)	Transversely Polarized (T)
Unpolarized (or Spin 0) Hadrons	D_1 = Unpolarized		$H_1^\perp$ = − Collins