Chapter 1

Learning Lessons For EFTs: Jelly

Beyond everything you learn in quantum field theory, effective field theories (EFTs) generically need three ingredients for formulate:

1. Relevant Degrees of Freedom
2. Symmetries
3. The State of Affairs and its Power Counting

The last point is most critical. You are specifying what is the physical state that you are examining, and the *weak* fluctuations about it. To get an idea of how these in points work, before we start using them in QCD, we will first examine jelly.

1.1 Jelly Elements

Jelly is at microscopic scales an amorphous entangled mass of proteins or starches, trapping sugar, water, and of course, flavor. We describe a jelly element by its departure from its initial position in the jelly. That is, if we are in a d-dimensional spacetime, we have scalar fields $\phi^I(\vec{x},t)$ with $I = 1, ..., d-1$. The vector $\vec{\phi}$ tells us where a jelly element went to at time t, if we take as our initial condition:

$$\vec{\phi}(\vec{x},0) = \vec{x}\,. \tag{1.1}$$

Thus $\vec{\phi}(\vec{x},t)$ tells us where the jelly element at $\vec{x}$ went at time t. These are our degrees of freedom.

1.2 Jelly Symmetries

What are the symmetries?

- Lorentz.
- Shifts $\vec{\phi} \to \vec{\phi} + \vec{a}$. How we labeled the jelly elements doesn't matter, we are interested in displacements.
- Rotations $\phi^I \to R^I_J \phi^J$, $R \in SO(d-1)$. At mesoscopic scales, jelly has no lattice structure, just a random jumble of knots.

The symmetries tell us how to organize our degrees of freedom into invariants. Our action for our degrees of freedom can only depend on the invariants, and physical observables can only depend on the invariants. For jelly, we then have the set:

$$B^{IJ} = \partial_\mu \phi^I \partial^\mu \phi^J \,. \tag{1.2}$$

$$X_1 = \mathrm{tr}[B]. \tag{1.3}$$

$$X_k = \frac{\mathrm{tr}[B^k]}{\mathrm{tr}[B]^{k-1}}, k = 2, ..., d-1. \tag{1.4}$$

All the X_k are independent of each other, and invariant under our above listed symmetries. So we can now write down our action:

$$S_{jelly} = \int d^d x F(X_1, X_2, ..., X_{d-1}) \,. \tag{1.5}$$

This action is valid for functions sufficiently smooth and bounded for the space of possible jelly configuration. Therefore it is useless. We have an infinite dimensional space of possible jelly actions, how are we going to perform enough experiments and gather enough data to nail them down? Silicon Valley would tell us to machine learn F after collecting several exabytes of data. I tell you need only to poke and jiggle.

1.3 Jelly Power Counting

We are not interested in the action for all jelly configurations. Indeed, most would be unphysical, involving so much energy that they would radically change the microscopic constituents. We care about small perturbations:

$$\vec{\phi}(\vec{x}, t) = \vec{x} + \vec{\pi}(\vec{x}, t), \qquad |\partial \vec{\pi}(\vec{x}, t)| \ll 1 \,. \tag{1.6}$$

Then we can expand our invariants:

$$X_k = a_k + b_k(\nabla \cdot \vec{\pi}) + c_k(\nabla \cdot \vec{\pi}) + d_k(\nabla \cdot \vec{\pi})^2 + e_k \sum_{I,J} \nabla^I \pi^J \nabla^I \pi^J + f_k(\partial_t \vec{\pi} \cdot \partial_t \vec{\pi}) + O\Big((\partial\pi)^3\Big) \tag{1.7}$$

the exact values for a_k, b_k etc. don't matter for us, it is straightforward to compute them, but suffice it to say that they are all different for different k. ∇ now refers to spatial derivatives, and ∂_t is the time derivative. Expanding our useless action Eq. (1.8) to second order in $\partial\pi$, and dropping constants and total derivatives, we have:

$$S_{jelly} = \int d^d x \left(\partial_t \vec{\pi} \cdot \partial_t \vec{\pi} - c_T^2 \sum_{I,J} \nabla^I \pi^J \nabla^I \pi^J \Big) - c_L^2 \Big(\nabla \cdot \vec{\pi} \Big)^2 \right) + O\Big((\partial\pi)^3\Big) \,. \tag{1.8}$$

We could calculate c_T and c_L in terms of the derivatives of F, after we calculate the $a_k, b_k, ...$ constants. What is more important is that there are just two of them for the independent structures in the action, after using freedom to rescale the action as we please. Thus we do not need advance neural net processors and learning computers to find F out of a huge dimensional space of possibilities. What are the physical low energy excitations? Writing:

$$\vec{\pi} = \vec{\pi}_T + \vec{\pi}_L \tag{1.9}$$

$$\nabla \cdot \pi = \nabla \cdot \pi_L \tag{1.10}$$

$$\nabla \cdot \pi_T = 0 \tag{1.11}$$

We have as our equations of motion:

$$\partial_t^2 \nabla \cdot \vec{\pi}_L = c_L^2 \nabla^2 \nabla \cdot \vec{\pi}_L \tag{1.12}$$

$$\partial_t^2 \vec{\pi}_T = c_T^2 \nabla^2 \vec{\pi}_T \tag{1.13}$$

That is, we have compressional and transverse waves. To fully understand jelly, as an eft, we need to only poke and jiggle it.

Note that symmetries and degrees of freedom alone were insufficient. We needed to also have an understanding of the state of affairs that we are trying to predict, with a notion of what it means to have a weak perturbation around this state of affairs.

Chapter 2

Lessons Learned For EFTs: QCD

2.1 Degrees of Freedom and Symmetries

We have already been introduced to the degrees of freedom for QCD, quarks and gluons. We have also been informed about its most important constraint on operators, gauge symmetry. But to be complete, we give a brief review. We have field operators and covariant derivatives given as:

$$\text{quarks: } \psi_i(x), \bar{\psi}_j(x)\,. \tag{2.1}$$
$$\text{gluons: } A^{A\mu}(x)\,. \tag{2.2}$$

These operators create and annilate quarks, anti-quarks, and gluons at space-time position x, where A, i, j are adjoint, fundamental and anti-fundamental representation indicies of the group $SU(N_c)$, μ is a lorentz index, and we have suppressed spinor indicies. $A^{A\mu}(x)$ is strangely behaved under gauge transformations, so we introduce the covariant derivative, acting on operators in the fundamental, anti-fundamental and adjoint respectively:

$$iD^\mu \psi_i(x) = i\partial^\mu \psi_i(x) + gA^{A\mu}(x)T^A_{ij}\psi_j(x) \tag{2.3}$$
$$iD^\mu \bar{\psi}_i(x) = i\partial^\mu \bar{\psi}_i(x) + gA^{A\mu}(x)T^A_{ji}\bar{\psi}_j(x) \tag{2.4}$$
$$iD^\mu \phi^A(x) = i\partial^\mu \phi^A(x) + gA^{B\mu}(x)f^{ABC}\phi^C(x) \tag{2.5}$$

Repeated indicies are summed. T^A_{ij} are the generator matricies of the Lie algebra in the fundamental representation, and f^{ABC} are the group's structure constants. Then the nicely behaved field for gluons is the field strength:

$$F^{A\mu\nu} = \mathrm{tr}([D^\mu, D^\nu]T^A)\,. \tag{2.6}$$

We act on the constant operator which is just the generator matricies of the Lie algebra in the fundamental representation, and take the trace over the fundmental indicies. Then we have the gauge transformations:

$$\psi_i(x) \to U_{ij}(x)\psi_j(x) \tag{2.7}$$
$$\bar{\psi}_j(x) \to \bar{\psi}_j(x)U^\dagger_{ji}(x) \tag{2.8}$$
$$F^{A\mu\nu}(x) \to \mathcal{U}^{AB}(x)F^{B\mu\nu}(x) \tag{2.9}$$
$$\mathcal{U}^{BA}(x)T^B_{ij} = U^\dagger_{i\ell}(x)T^A_{\ell k}U^\dagger_{kj}(x) \tag{2.10}$$

$U(x) \in SU(N_c)$, and operates in the fundamental representation, $\mathcal{U}$ is the same group element mapped to the adjoint. In what follows, we will often suppress the gauge group indicies. What is critical about the covariant derivative, is that it commutes through a gauge tranformation:

$$iD^\mu\psi(x) \to U(x)iD^\mu\psi = iD^\mu U(x)\psi(x), \tag{2.11}$$
$$iD^\alpha F^{\mu\nu}(x) \to iD^\alpha \mathcal{U}(x)F^{\mu\nu}(x) = \mathcal{U}(x)iD^\alpha F^{\mu\nu}(x) \tag{2.12}$$

2.2 Invariants

In QFT courses, you would then see the construction of *local* invariants, like:

$$F^A_{\mu\nu}F^{A\mu\nu}, \bar{\psi}\psi, \bar{\psi}T^A\sigma^{\mu\nu}\psi F^A_{\mu\nu}, \tag{2.13}$$

For constructing an eft for high energy scattering processes, local invariants will not cut it. We need non-local invariants. We will see why in a moment when we examine some scattering processes, but for now, let us build non-local invariants.

To make something non-local, we need to be able to move it in space-time. To move something in space-time, we need to take its derivative. To move something in space-time without changing its gauge transformation properties, we need a covariant derivative. Let dx be a differential step in space-time, and $\phi(x)$ an operator in some fixed representation, which does not change according to the covariant derivative when we step in dx:

$$dx^\mu iD_\mu\phi(x) = 0 \tag{2.14}$$

Then we can expand:

$$\begin{aligned} \phi(x+dx) &= \phi(x) + dx\cdot\partial\phi(x) + ... && (2.15)\\ &= \phi(x) - igdx\cdot A^A(x)T^A\phi(x) + ... && (2.16)\\ &= e^{-igdx\cdot A^A(x)T^A}\phi(x) + ... && (2.17) \end{aligned}$$

Under a gauge transformation:

$$U(x+dx)\phi(x+dx) = U(x+dx)e^{-igdx\cdot A^A(x)T^A}\phi(x) + ... \tag{2.18}$$

But $\phi(x) \to U(x)$, so we have to have:

$$e^{-igdx\cdot A^A(x)T^A} \to U(x+dx)e^{-igdx\cdot A^A(x)T^A}U^\dagger(x) + ... \tag{2.19}$$

That is, the operator $e^{-igdx\cdot A^A(x)T^A}$ moves the gauge transformation over a step dx.

Now if we start at x_i, and want to end at x_f, then we need to specify a path Γ broken into k differential steps $dx_1, dx_2, ..., dx_k$, and consider the operator:

$$S^{[k]}_\Gamma(x_f, x_i) = e^{-igdx_k\cdot A^A(x_i+\sum_{\ell=1}^{k-1}dx_\ell)T^A}e^{-igdx_{k-1}\cdot A^A(x_i+\sum_{\ell=1}^{k-2}dx_\ell)T^A}...e^{-igdx_1\cdot A^A(x_i)T^A}\,. \tag{2.20}$$

As $k\to\infty$, we get the path-ordered exponential:

$$\begin{aligned} S_\Gamma(x_f, x_i) &= \lim_{k\to} e^{-igdx_k\cdot A^A(x_i+\sum_{\ell=1}^{k-1}dx_\ell)T^A}e^{-igdx_{k-1}\cdot A^A(x_i+\sum_{\ell=1}^{k-2}dx_\ell)T^A}...e^{-igdx_1\cdot A^A(x_i)T^A}\,. && (2.21)\\ &= \mathrm{Pexp}\Big(-ig\int_\Gamma dx\cdot A^A(x)T^A\Big) && (2.22) \end{aligned}$$

These are wilson lines, and have the following gauge transformation properties:

$$S_\Gamma(x_f, x_i) \to U(x_f)S_\Gamma(x_f, x_i)U^\dagger(x_i)\,. \tag{2.23}$$

So we can build *non-local* invariants, like:

$$\bar{\psi}(x_f)S_\Gamma(x_f, x_i)\psi(x_i)\,. \tag{2.24}$$

For us, given a lorentz vector v, we will be particularly interested in wilson line operators of the form:

$$\begin{aligned} S_v(x; t_f, t_i) &= \mathrm{Pexp}\Big(-ig\int_{t_i}^{t_f}dt'v\cdot A^A(vt'+x)T^A\Big) && (2.25)\\ &= 1 - ig\int_{t_i}^{t_f}dt'v\cdot A^A(vt'+x)T^A + (-ig)^2\int_{t_i}^{t_f}dt_1\int_{t_i}^{t_1}dt_2\Big(v\cdot A^A(vt_1+x)T^A\Big)\Big(v\cdot A^B(vt_2+x)T^B\Big) + ... && (2.26) \end{aligned}$$

We note that such wilson lines enjoy the property:

$$v \cdot DS_v(x;t_f,t_i) = 0 \tag{2.27}$$

wilson-lines are always a matrix in some particular representation. We will often use the short-hand for the semi-infinite wilson line:

$$S_v(x) = S_v(x;\infty,0)\,. \tag{2.28}$$

In covariant gauges, these wilson lines have no gauge transformation at infinity. More generally, this is a subtle point, but for our purposes, we have:

$$S_v(x) \to S_v(x)U^\dagger(x)\,. \tag{2.29}$$

This means, making gauge indicies explicit, we can form invariant objects like:

$$T\{S_{v_1}^{ij}(x)S_{v_2}^{\dagger jk}(x)\}. \tag{2.30}$$

This is a cusped wilson line,

2.3 State of Affairs

When we are analyzing drell-yan, or higgs production or decay, or say back to back hadron production in e^+e^-, a particular state of affairs that we are interested in is a two jet configuration. So let us consider the decay of a higgs boson at tree level to two gluons. This will be our "ground state" that we wish to understand weak perturbations around. What do we mean by weak perturbations? We mean any additional radiation that keeps the same overall momentum flow as our two gluon decay. Thus any additional radiation must soft (i.e., not energetic, but going anyway) or collinear to one of the gluons considered. Our amplitude for this is:

$$\mathcal{A}\Big(q_1\epsilon_1^{A_1}, q_2\epsilon_2^{A_2}\Big) \propto N\Big(q_1\cdot q_2\epsilon_1\cdot\epsilon_2 - \epsilon_2\cdot q_1\epsilon_1\cdot q_2\Big) \tag{2.31}$$

Where q_i, ϵ_i, A_i is the momentum, polarization, and color index of the i-th gluon, and $q_1 \cdot q_2$ is large. Now we consider the 3-gluon amplitude, also at tree level:

$$\mathcal{A}\Big(p_1\epsilon_1^{A_1}, p_2\epsilon_2^{A_2}, p_3\epsilon_3^{A_3}\Big) \tag{2.32}$$

We wish to take the soft limit:

$$p_1 \to q_1, p_2 \to q_2, \frac{q_1\cdot p_3}{q_1\cdot q_2} \to 0, \frac{q_2\cdot p_3}{q_1\cdot q_2} \to 0 \tag{2.33}$$

It is straightforward calculation to find:

$$\lim_{p_3 \text{ soft}} \mathcal{A}\Big(p_1\epsilon_1^{A_1}, p_2\epsilon_2^{A_2}, p_3\epsilon_3^{A_3}\Big) = -ig\mathcal{A}\Big(q_1\epsilon_1^{S_1}, q_2\epsilon_2^{S_2}\Big)\left(\frac{q_1\cdot\epsilon_3}{q_1\cdot p_3}\delta^{S_2A_2}f^{A_1S_1A_3} + \frac{q_2\cdot\epsilon_3}{q_2\cdot p_3}\delta^{S_1A_1}f^{A_2S_2A_3}\right) + \ldots \tag{2.34}$$

Repeated indicies are summed. Now we ask what gauge invariant operators interpolate the same result?

$$\lim_{p_3 \text{ soft}} \mathcal{A}\Big(p_1\epsilon_1^{A_1}, p_2\epsilon_2^{A_2}, p_3\epsilon_3^{A_3}\Big) = N\langle 0|\hat{O}_{hc}|q_1\epsilon_1^{A_1}, q_2\epsilon_2^{A_2}\rangle \otimes \langle 0|\hat{O}_s|p_3\epsilon_3^{A_3}\rangle \tag{2.35}$$

The $\otimes$ state for various possible contractions and convolutions between the soft and hard-collinear operators wee have written down, as the two must be correlated in directions and color as we can see from the expansion above.

Also, it is clear from the expansion above that the soft operator $\hat{O}_s$ must be non-local, because our expansion of the tree-level result for the three-point amplitude was already not polynomial in the momentum of p_3. Semi-classically, the wilson lines introduced above represents a charge moving on a world-line radiating. Using the expansion in Eq. (2.26) above, it is simple to see:

$$\langle 0|T\{S_{q_1}^{S_1A_1}(0)S_{q_2}^{S_2A_2}(0)\}|p_3\epsilon_3^{A_3}\rangle = -ig\left(\frac{q_1\cdot\epsilon_3}{q_1\cdot p_3}\delta^{S_2A_2}f^{A_1S_1A_3} + \frac{q_2\cdot\epsilon_3}{q_2\cdot p_3}\delta^{S_1A_1}f^{A_2S_2A_3}\right) \tag{2.36}$$

Thus the operator $\hat{O}_s = T\{S_{q_1}^{S_1 A_1}(0) S_{q_2}^{S_2 A_2}(0)\}$ fulfills our task, as long as the indicies S_1 and S_2 get contracted into an $SU(N_c)$ invariant tensor. But not only that, this operator which we found by looking for gauge invariant operators whose matrix elements reproduce the lowest order perturbations about our "groud state" will generalize to interpolate all possible soft radiation around our two hard gluon states. Once we flesh out the power-counting, we will find that any other operator (up to Glauber exchanges!) we might one to use to do this will be power suppressed! This operator will be unique in this regard: if the additional soft radiation does factorize from the hard configuration, it must be described by matrix elements of $T\{S_{q_1}^{S_1 A_1}(0) S_{q_2}^{S_2 A_2}(0)\}$.

2.3.1 Hard-Collinear Operators and Power Counting

Up till now, we just looked for a non-local operator that was easily made gauge invariant which reproduced our soft radiation. We now need to examine our hard-collinear contribution:

$$\langle 0|\hat{O}_{hc}|q_1 \epsilon_1^{A_1}, q_2 \epsilon_2^{A_2}\rangle \tag{2.37}$$

What will interpolate our hard collinear states? Field operators! But which ones? Seemingly, any will do. Again, to narrow the possibilities, we need to use power counting. So we introduce the following definition. If we have light-cone decomposition of a momentum p:

$$\begin{aligned}
&n^2 = \bar{n}^2 = 0, \quad n \cdot \bar{n} = 2, \text{ and } \perp \text{ denotes all components out of the plane formed from } n, \bar{n} \\
&p^\mu = \bar{n} \cdot p \frac{n^\mu}{2} + n \cdot p \frac{\bar{n}^\mu}{2} + p_\perp^\mu
\end{aligned} \tag{2.38}$$

The momenta $p_1, ..., p_k$ are *collinear* to null direction n if there is a $\lambda \ll 1$ and a mass scale Q, such

$$O(\bar{n} \cdot p) = O(Q), \quad O(n \cdot p) = O(Q\lambda^2), \quad O(p_\perp) = O(Q\lambda) \tag{2.39}$$

Then note $p_i \cdot p_j \sim Q^2 \lambda^2$, and

$$p_i^2 = n \cdot p_i \bar{n} \cdot p_i - p_{i\perp}^2 \sim Q^2 \lambda^2 \tag{2.40}$$

for each i. Critically, all the terms in the light-cone decomposition of p_i^2 or $p_i \cdot p_J$ scale homogenously. In these coordinates there is nothing to expand.