Lorentz Invariance relations and Equation of Motion Relations for polarized GTMDs

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Objectives

1. *GTMDs as functions of GPCFs*
2. *Lorentz Invariance Relations*
3. *Equation of Motion Relations*

GTMDs as functions of GPCFs

GPCFs considered as the mother distribution function of GTMDs and GPDs.

$$W^{\Gamma}_{\Lambda'\Lambda}(P,K,\Delta;u) = \frac{1}{2}\int \frac{d^4z}{(2\pi)^4} e^{iK\cdot z}\langle p',\Lambda'|\overline{\psi}(-\frac{z}{2})\Gamma u\psi(\frac{z}{2})|p,\Lambda\rangle$$

- The GTMDs limit K^-- integration

$$W^{\Gamma}_{\Lambda'\Lambda}(P,x,K_T,\xi,\Delta_T;u) = \int dK^- W^{\Gamma}_{\Lambda'\Lambda}(P,K,\Delta;u)$$
$$= \frac{1}{2}\int \frac{dz^- d^2z_T}{(2\pi)^3} e^{ixp^+z^- - iK_T\cdot z_T}\ \langle p',\Lambda'|\overline{\psi}(-\frac{z}{2})\Gamma u\psi(\frac{z}{2})|p,\Lambda\rangle|_{z^+=0}$$

- GPDs-limit-K_T integral

$$F^{\Gamma}_{\Lambda'\Lambda}(x,\xi,t) = \int dK^- d^2K_T W^{\Gamma}_{\Lambda'\Lambda}(P,K,\Delta;u)$$

$$= \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixp^+z^-} \langle p',\Lambda'|\overline{\psi}(-\frac{z}{2})\Gamma u \psi(\frac{z}{2})|p,\Lambda\rangle|_{z^+=0,\ z_T=0}$$

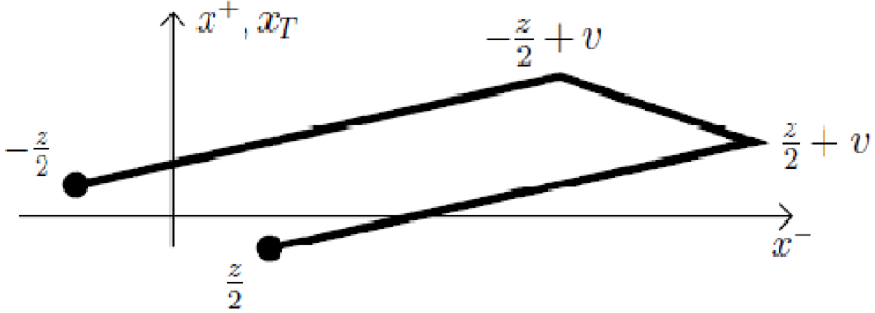
$\Gamma = 1, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, i\sigma^{\mu\nu}$

$$P = \left[\ p^+, \frac{\vec{\Delta}_T^2 + 4M^2}{8(1-\xi^2)P^+}\ ,\ \vec{0}_T\ \right]$$

$$K = \left[\ xp^+, K^-\ , \vec{K}_T\ \right]$$

$$\Delta = \left[-2\xi p^+, \frac{\xi\vec{\Delta}_T^2 + 4\xi M^2}{4(1-\xi^2)p^+},\ \vec{\Delta}_T\ \right]$$

Stable-shaped gauge link path connecting quark operators located at -z/2 and z/2. GTMDs are defined at sparation $z^+ = 0$. and $v = (0, v^-, 0, 0)$



The correlations in terms of GPCFs amplitudes (A)

$$W^{\gamma^{\mu}}_{\Lambda'\Lambda} = \overline{U}(p', \Lambda')\Big[\frac{P^{\mu}}{M}A_1^F + \frac{K^{\mu}}{M}A_2^F + \frac{\Delta^{\mu}}{M}A_3^F + \frac{i\sigma^{\mu K}}{M}A_5^F + \frac{i\sigma^{\mu\Delta}}{M}A_6^F + \frac{i\sigma^{K\Delta}}{M^2}\Big(\frac{P^{\mu}}{M}A_8^F + \frac{K^{\mu}}{M}A_9^F + \frac{\Delta^{\mu}}{M}A_{17}^F\Big)\Big]U(p, \Lambda)$$

For the axial vector $\Gamma = \gamma^{\mu}\gamma^5$,

$$W^{\gamma^{\mu}\gamma^5}_{\Lambda'\Lambda} = \overline{U}(p', \Lambda')\Big[\frac{i\epsilon^{\mu PK\Delta}}{M^3}A_1^G + \frac{i\sigma^{P\mu}\gamma^5}{M}A_{17}^G + \frac{i\sigma^{PK}\gamma^5}{M^2}\Big(\frac{P^{\mu}}{M}A_{18}^G + \frac{K^{\mu}}{M}A_{19}^G + \frac{\Delta^{\mu}}{M}A_{20}^G\Big) + \frac{i\sigma^{P\Delta}\gamma^5}{M^2}\Big(\frac{P^{\mu}}{M}A_{21}^G + \frac{K^{\mu}}{M}A_{22}^G + \frac{\Delta^{\mu}}{M}A_{23}^G\Big)\Big]U(p, \Lambda)$$

- All GPCFs are functions of $K^2, K.P, K.\Delta, \Delta^2$, and$P.\Delta$

Twist-(two,three) vector GTMDs

$$W^{\gamma^+}_{\Lambda'\Lambda} = \frac{1}{2M}\overline{U}(p^{'},\Lambda^{'})\left[F_{11} + \frac{i\sigma^{i+}K^i}{p^+}F_{12} + \frac{i\sigma^{i+}\Delta^i}{p^+}F_{13} + \frac{i\sigma^{ij}K^i\Delta^j}{M^2}F_{14}\right]U(p,\Lambda)$$

- No power of p^+- twist two

$$W^{\gamma^i}_{\Lambda'\Lambda} = \frac{1}{2p^+}\overline{U}(p^{'},\Lambda^{'})\left[\frac{K^i}{M}F_{21} + \frac{\Delta^i}{M}F_{22} + \frac{Mi\sigma^{i+}}{p^+}F_{23} + \frac{K^i i\sigma^{K+}K^K}{Mp^+}F_{24} + \frac{\Delta^i i\sigma^{K+}K^k}{Mp^+}F_{25} + \frac{\Delta^i i\sigma^{k+}\Delta^K}{Mp^+}F_{26} + \frac{i\sigma^{ji}K^j}{M}F_{27} + \frac{i\sigma^{ji}\Delta^j}{M}F_{28}\right]U(p,\Lambda)$$

- suppression by one power of p^+-Twist-three

Matching the vector case ($\Gamma = \gamma^\mu$) correlation with the GTMDs correlations.

- Twist-two vector GTMDs as functions of GPCFs

$$F_{11} = 2p^+ \int dK^- \Big[A_1^F + xA_2^F - 2\xi A_3^F + x\xi A_5^F - 2\xi^2 A_6^F - \left(\frac{2\xi^2 P\cdot K}{M^2} + \frac{x\Delta_T^2}{2M^2} + \frac{\xi K_T\cdot\Delta_T}{M^2} \right)\left(A_8^F + xA_9^F - 2\xi A_{17}^F \right) \Big]$$

$$F_{12} = 2p^+ \int dK^- \left[A_5^F - \frac{2\xi P^2}{M^2}\left(A_8^F + xA_9^F - 2\xi A_{17}^F \right) \right]$$

$$F_{13} = 2p^+ \int dK^- \left[A_6^F + \frac{P\cdot K - xP^2}{M^2}\left(A_8^F + xA_9^F - 2\xi A_{17}^F \right) \right]$$

$$F_{14} = 2p^+ \int dK^- \left[A_8^F + xA_9^F - 2\xi A_{17}^F \right]$$

- Twist-three vector GTMDs as functions of GPCFs

$$F_{21} = 2p^+ \int dK^- \left[A_2^F - \frac{4\xi^2 P\cdot K + x\boldsymbol{\Delta}_T^2 + 2\xi(K_T\cdot\boldsymbol{\Delta}_T)}{2M^2} A_9^F \right]$$

$$F_{22} = 2p^+ \int dK^- \left[A_3^F - \frac{x}{2}A_5^F + \xi A_6^F - \frac{4\xi^2 P\cdot K + x\boldsymbol{\Delta}_T^2 + 2\xi(K_T\cdot\boldsymbol{\Delta}_T)}{2M^2} A_{17}^F \right]$$

$$F_{23} = 2p^+ \int dK^- \left[\frac{P\cdot K - xP^2}{M^2}\left(A_5^F + \frac{(K_T\cdot\boldsymbol{\Delta}_T)^2 - K_T^2\boldsymbol{\Delta}_T^2}{M^2(K_T\cdot\boldsymbol{\Delta}_T)} A_9^F \right) + \frac{2\xi P^2}{M^2}\left(A_6^F + \frac{(K_T\cdot\boldsymbol{\Delta}_T)^2 - \boldsymbol{\Delta}_T^2 K_T^2}{M^2(K_T\cdot\boldsymbol{\Delta}_T)} A_{17}^F \right) \right]$$

$$F_{24} = 2p^+ \int dK^- \left[\left(\frac{P\cdot K - xP^2}{M^2}\frac{\boldsymbol{\Delta}_T^2}{(K_T\cdot\boldsymbol{\Delta}_T)} - \frac{2\xi P^2}{M^2} \right) A_9^F \right]$$

$$F_{25} = 2p^+ \int dK^- \left[\frac{xP^2 - P\cdot K}{M^2} A_9^F - \frac{2\xi P^2}{M^2} A_{17}^F \right]$$

$$F_{26} = 2p^+ \int dk^- \left[\frac{P\cdot K - xP^2}{M^2}\left(\frac{K_T^2}{(K_T\cdot\boldsymbol{\Delta}_T)} A_9^F + A_{17}^F \right) \right]$$

$$F_{27} = 2p^+ \int dK^- \left[A_5^F + \frac{\boldsymbol{\Delta}_T\cdot K_T}{M^2} A_9^F + \frac{\boldsymbol{\Delta}_T^2}{M^2} A_{17}^F \right]$$

$$F_{28} = 2p^+ \int dK^- \left[A_6^F - \frac{K_T^2}{M^2} A_9^F - \frac{\boldsymbol{\Delta}_T\cdot K_T}{M^2} A_{17}^F \right]$$

Lorentz Invariance Relations

LIRs are relations that connect the x-dependent K_T—moments of GTMDs with the GPDs.

- K_T- moments of GTMDs

$$X^{(1)} = \frac{2}{M^2} \int d^2 K_T \frac{K_T^2 \Delta_T^2 - (K_T \cdot \Delta_T)^2}{\Delta_T^2} X(x, \xi, K_T^2, K_T \cdot \Delta_T, \Delta_T^2)$$

- Scalar quantities

$$\sigma \equiv 2(P \cdot K) = 2p^+ K^- + 2p^- K^+ = 2p^+ K^- + xp^2 \Longrightarrow K^- = \frac{1}{2P^+}(\sigma - xP^2)$$

$$\tau \equiv K^2 = 2K^+ K^- - K_T^2 = x\sigma - x^2 P^2 - K_T^2$$

$$\sigma' \equiv K \cdot \Delta = K^+ \Delta^- + K^- \Delta^+ - K_T \cdot \Delta_T = 2\xi x P^2 - \xi\sigma - K_T \Delta_T \cos\varphi$$

$$\frac{d}{dx}\int d^2K_T\int dK^-\frac{K_T^2\Delta_T^2-(K_T\cdot\Delta_T)^2}{\Delta_T^2}\chi[A;x]$$

$$=\int d^2K_T\int dK^-\left(P\cdot K-xP^2-\frac{2\xi p^2(K_T\cdot\Delta_T)}{\Delta_T^2}\right)\chi[A;x]$$

$$+\int d^2K_T\int dK^-\frac{K_T^2\Delta_T^2-(K_T\cdot\Delta_T)^2}{\Delta_T^2}\frac{d}{dx}\chi[A;x]$$

- LIRs

$$\frac{dF_{12}^{(1)}}{dx}=-2\left[\frac{\xi p^2}{M^2}\left(E+H\right)+H_{2T}+\frac{\xi p^2}{M^2}\left(\widetilde{E}_{2T}-\xi E_{2T}\right)\right]$$

$$\frac{dG_{12}^{(1)}}{dx}=2\left[H_{2T}'-\frac{4\xi^2p^2+\Delta_T^2}{4M^2}E_{2T}'-\frac{p^2}{M^2}\widetilde{H}+\frac{\xi p^2}{M^2}\widetilde{E}_{2T}'\right]$$

Equation of Motion Relations

Defining the equation of motion to the GTMD correlator W

$$(i\not{D} - m)\psi = (i\not{\partial} + g\not{A} - m)\psi = 0$$

$$\overline{\psi}(i\overleftarrow{\not{D}} + m) = \overline{\psi}(i\overleftarrow{\not{\partial}} - g\not{A} + m) = 0$$

- Integrate by parts, one gets Relations between GTMD correlators, e.g.,

$$-\frac{\Delta^+}{2}W^{\gamma^i\gamma^5}_{\Lambda'\Lambda} + iK^+\epsilon^{ij}W^{\gamma^j}_{\Lambda'\Lambda} + \frac{\Delta^i}{2}W^{\gamma^+\gamma^5}_{\Lambda'\Lambda} - i\epsilon^{ij}K^j W^{\gamma^+}_{\Lambda'\Lambda} + M^{i,S}_{\Lambda'\Lambda} = 0$$

- $M^{i,S}_{\Lambda'\Lambda}$ is the qgq term. For straight gauge link

$$\int dx \int d^2K_T M^{i,S}_{\Lambda'\Lambda} = o$$

Consider the combination $(\Delta_1 + i\Delta_2)W^{\Gamma}_{+-} + (\Delta_1 - i\Delta_2)W^{\Gamma}_{-+}$, contracted with Δ^i, to the last equation gives,

$$\left[(1-\xi^2)H'_{2T} + \xi\widetilde{E}'_{2T} - \xi^2 E'_{2T}\right] + x\left[\frac{(1-\xi^2)}{\xi}H_{2T} - \xi E_{2T} + \widetilde{E}_{2T}\right] + \frac{\Delta_T^2}{4M^2}\widetilde{E}$$

$$+ \frac{\Delta_T^2}{4M^2}F_{14}^{(1)} + \frac{(1-\xi^2)}{2\xi}F_{12}^{(1)}$$

$$+ \frac{\Delta^i\sqrt{1-\xi^2}}{2\xi M^2\Delta_T^2}\int K_T^2[(\Delta_1 - i\Delta_2)M^{i,S}_{-+} + (\Delta_1 + i\Delta_2)M^{i,S}_{-+}] = 0$$

- Taking x-derivative.
- Eliminating GTMDs by LIRs.

$$\frac{dF_{12}^{(1)}}{dx} = -2\left[\frac{\xi p^2}{M^2}\left(E + H\right) + H_{2T} + \frac{\xi p^2}{M^2}\left(\widetilde{E}_{2T} - \xi E_{2T}\right)\right]$$

$$\frac{dF_{14}^{(1)}}{dx} = \left[E + H + \widetilde{E}_{2T} - \xi E_{2T}\right]$$

- At the limit $\Delta \longrightarrow 0$

$$\frac{1}{2}\int dxx\Big(E+H\Big) = \int dxx\Big(\widetilde{E}_{2T}+E+H+H_{2T\xi}\Big) + \frac{1}{2}\int dxH'_{2T}$$

where

$$H_{2T\xi} = \lim_{\xi\rightarrow 0}\frac{1}{\xi}H_{2T}$$

$$J = L + S$$

- Operators
- $M^{\mu\nu}$: generalized angular momentum tensor.
- M^{jk} spatial component
- K : boost operator.
- $K^{i} = M^{0i}$ pure Lrentz transformation along x, y and z axes.
- $[J^{i}, K^{j}] = i\epsilon^{ijk}K^{K}$

Use Lorentz transformation properties of (0 i) components of rank-2 tensor for longitudinal/transverse i

$$J_T = \gamma J_0 = \gamma J_L = \overbrace{\left(P^+ \frac{1}{\sqrt{2}M} + \frac{1}{P^+}\frac{M}{2\sqrt{2}}\right)}^{\gamma} \overbrace{\frac{1}{2}\int dx x (H+E)}^{J_L}$$

in the large-momentum limit, angular momentum, orbital angular momentum, and spin can be expanded in powers of $1/P^+$ as

$$J_T = J_T^{(1)} P^+ + J_T^{(-1)} \frac{1}{P^+} + \ldots$$

$$L_T = L_T^{(1)} P^+ + L_T^{(-1)} \frac{1}{P^+} + \ldots$$

$$S_T = S_T^{(1)} P^+ + S_T^{(-1)} \frac{1}{P^+} + \ldots$$

· Transverse Spin

$$S_T^{(1)} = 0$$

$$S_T^{(-1)} = \frac{M}{\sqrt{2}}\frac{1}{2}\int dx g_T = \frac{M}{\sqrt{2}}\frac{1}{2}\int dx H_{2T}^{'}|_{\Delta=0}$$

Multiply the equation of motion by $M/\sqrt{2}$

$$\frac{1}{2}\int dx x\Big(E + H\Big) = \int dx x\Big(\widetilde{E}_{2T} + E + H + H_{2T\xi}\Big) + \frac{1}{2}\int dx H_{2T}'$$

$$\frac{M}{2\sqrt{2}}\left[\frac{1}{2}\int dxx\Big(E+H\Big)\right] =$$

$$\frac{M}{\sqrt{2}}\left[\int dxx\Big(\widetilde{E}_{2T}+E+H+H_{2T\xi}\Big)\right]$$

$$-\frac{M}{2\sqrt{2}}\left[\frac{1}{2}\int dxx\Big(E+H\Big)\right]$$

$$+\frac{M}{\sqrt{2}}\left[\frac{1}{2}\int dxH'_{2T}\right]$$

$$J_T^{(-1)} = \frac{1}{2}\int dx x\Big(E + H\Big)$$

$$L_T^{(-1)} = \int dx x\Big(\widetilde{E}_{2T} + E + H + H_{2T\xi}\Big) = \frac{1}{2}\lim_{\xi\to 0}\int dx \frac{F_{12}^{(1)}}{\xi}$$

$$S_T^{(-1)} = \frac{1}{2}\int dx H'_{2T} = \frac{1}{2}\int dx\Big(g_1 + g_2\Big)$$

- The distributions are taken in the forward limit $\Delta = 0$.
- The straight link is used.