Lattice-to-continuum factorization for TMDs

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TMD Collaboration Meeting, Santa Fe

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Support:

Based on:
2004.14831
2201.08401
2205.12369

You’ve already seen many TMD talks…

So I can breeze through the usual intro.

The parton model…

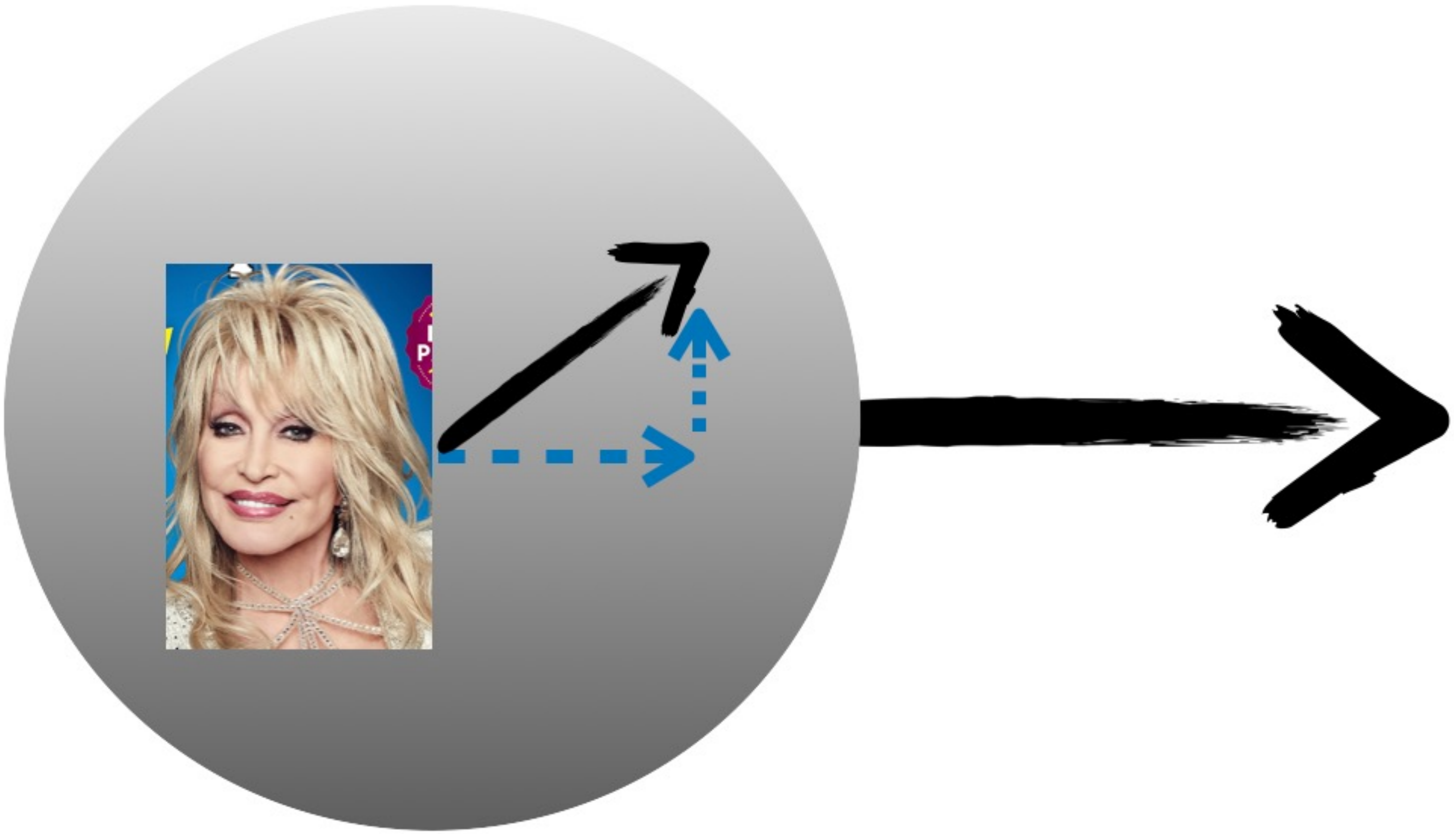
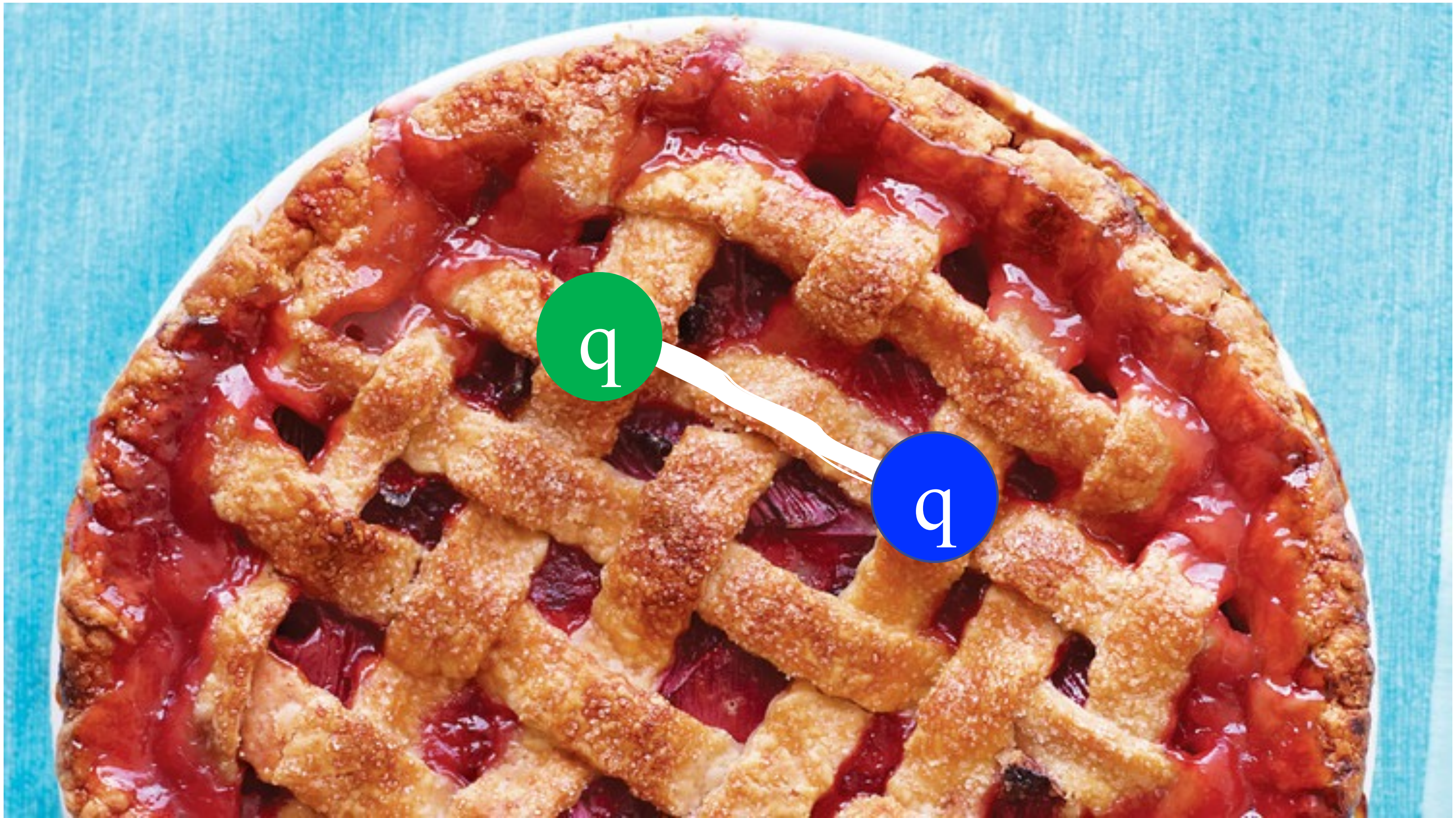


Figure: *People* magazine.

Introduction to lattice

Figure: Martha Stewart's website.



What “TMD” means

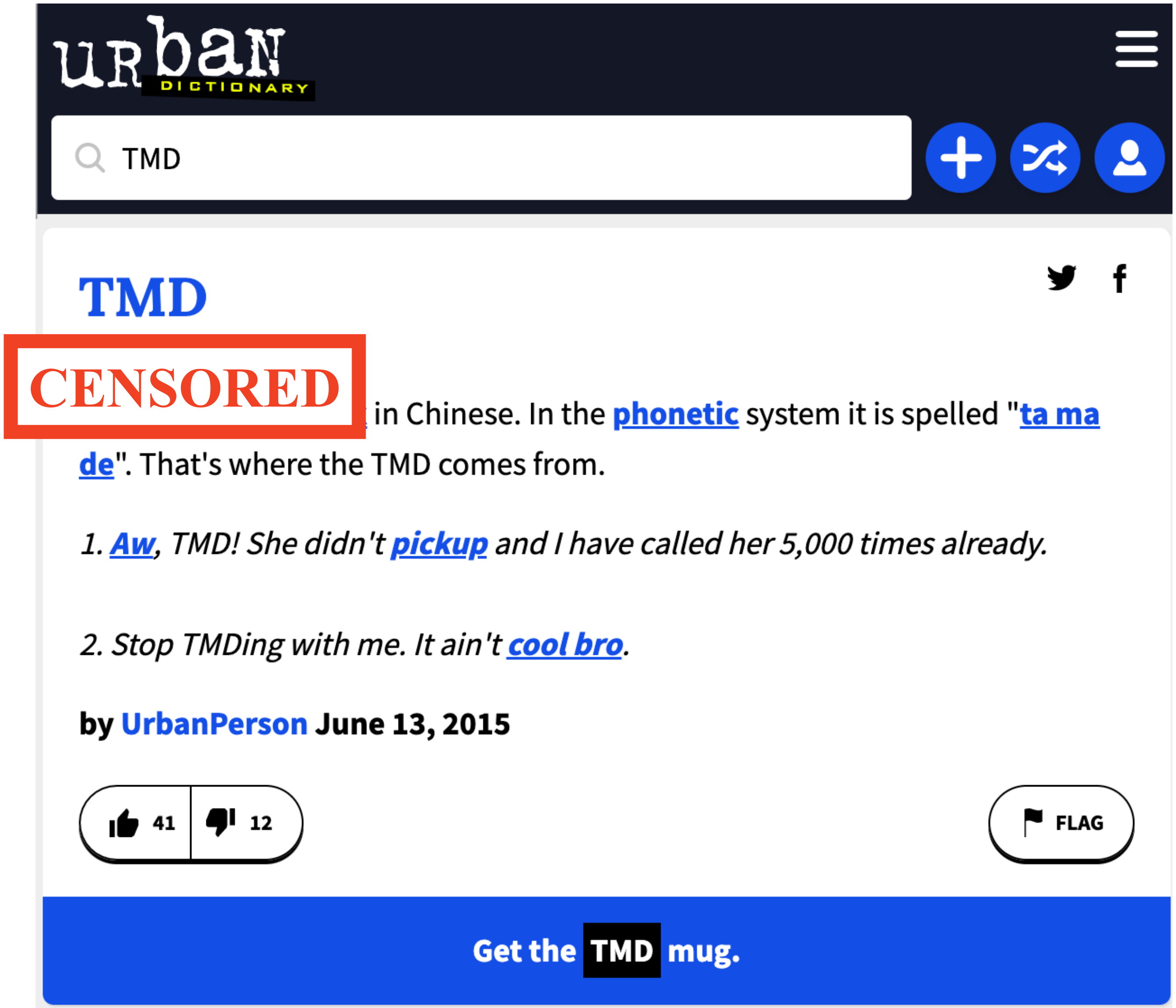
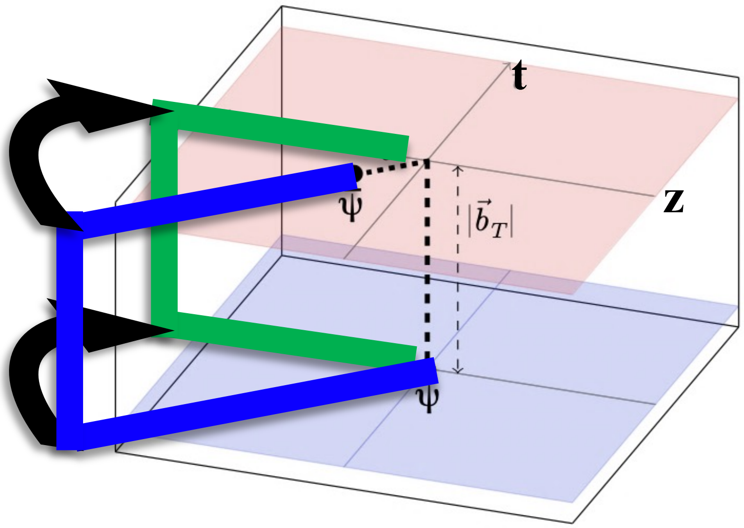


Figure: Urban Dictionary.

Motivation:

Non-perturbative contributions to TMDs, from first principles

You can't put a TMD on the lattice directly



Defined by **lightcone Wilson lines**:

- Dependent on time variable
- Naïve discretization → real-time "sign problem"
- Prohibitive computational cost!

Instead, calculate TMDs indirectly:

1. **Projection:** **time-dependent** ➔ **equal-time Wilson line**
2. **Factorization:** formula relating **physical** & **lattice** TMDs

Three key ingredients

1. Numerically tractable “Lattice TMDs”

2. Precision lattice calculations

3. **Connection to physical TMDs**

TMD factorization

Experimental data

(e.g. Drell-Yan process)

$$d\sigma = H \int f \otimes f$$

Renormalized continuum QCD

$$f = Z_{UV} \frac{B}{\sqrt{S}}$$

Goal

Lattice-regularized QCD

$$f = C \times \tilde{f}_{lattice}$$

Collins, *Foundations of Perturbative QCD*. Ebert, **Schindler**, Stewart, and Zhao (JHEP 2022).

Outline of today's lecture

	Me	You
I. Historical overview	🤔	🙂
II. New notation	🤓	🥱
III. Factorization	😎	😱
IV. Outlook	👀	🥳

Historical overview of lattice TMDs

2013 — **First lattice TMD proposed: MHENS scheme**
Musch, Hägler, Engelhardt, Negele, and Schäfer

2014 — **New lattice scheme proposed (quasi), 1-loop calculations**
Xiangdong Ji

⋮ — **Lattice calculations of MHENS beam functions**
MHENS and collaborators

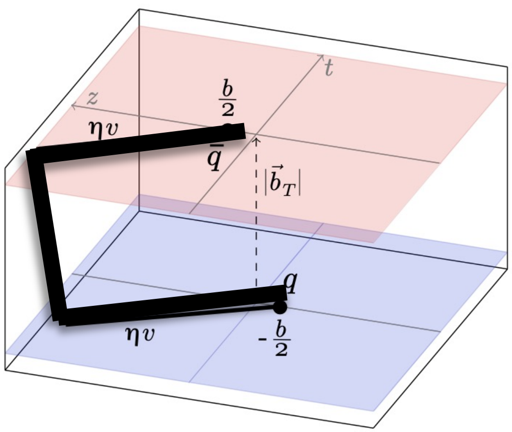
2018 — **Theory of quasi-TMDs put on firmer footing**
Ebert, Stewart, and Zhao

2019 — **Proposal for lattice calculation of quasi-soft function**
Ji, Liu, and Liu

2020-22 — **First lattice results for CS kernel & quasi-soft function**
MIT, LPC, ETMC, and Regensburg lattice groups

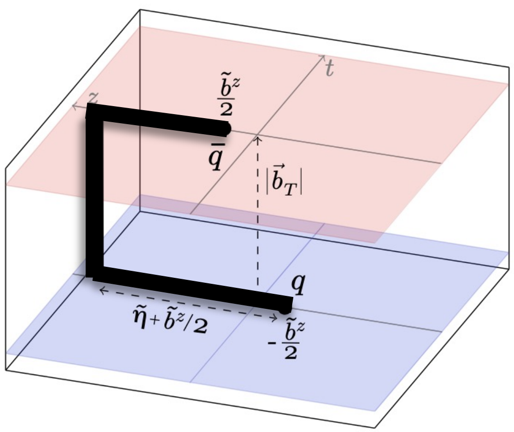
Two main lattice approaches

MHENS scheme



- Pioneered lattice TMDs
- Focused on *x*-moments
- Renormalization, soft function not fully known

Quasi-TMDs

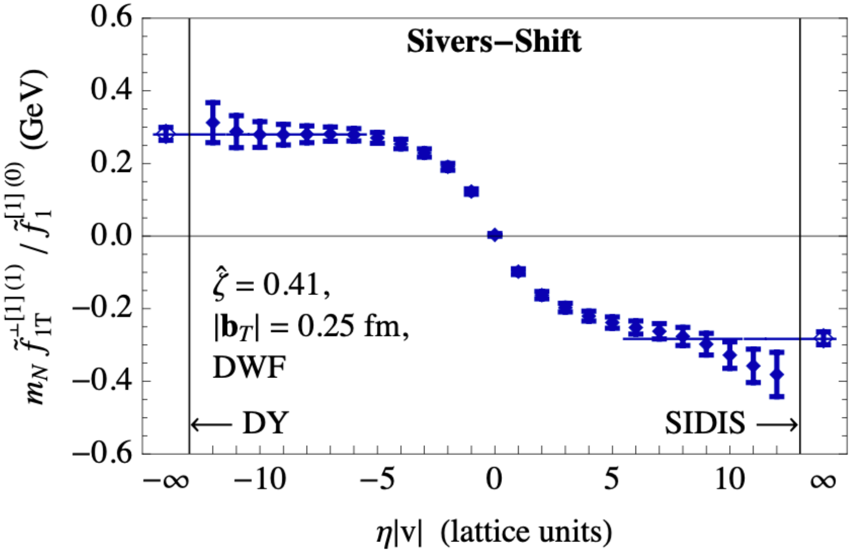


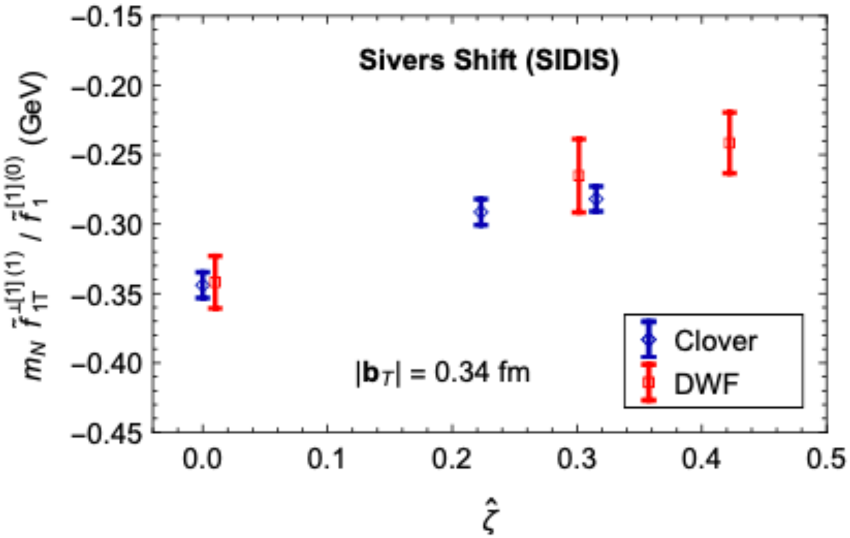
- Newer; fewer results for proton
- Focused on full TMD
- Renormalization, soft function have been proposed

MHENS on the lattice

Example: sign change of the Sivers function in SIDIS & Drell-Yan:

[Yoon, Engelhardt, Gupta, et al. (PRD 2017).]





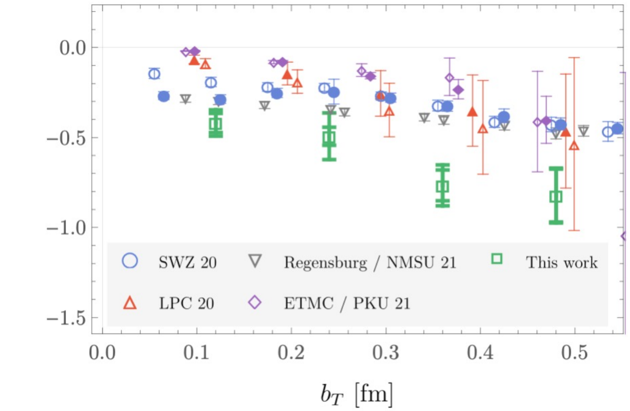
Many observables have been studied!

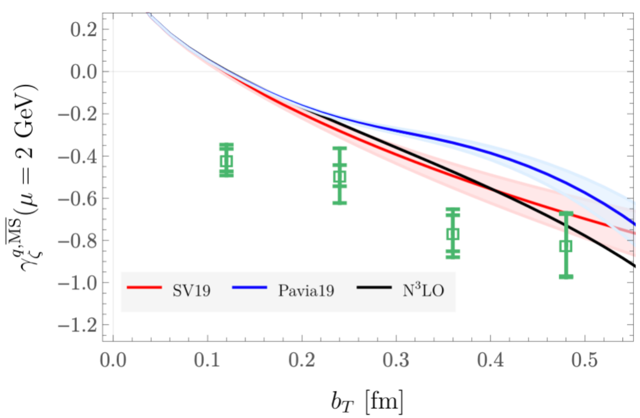
Quasi-TMDs on the lattice

Recent first calculations of **<u>all</u>** TMD components!

CS Kernel

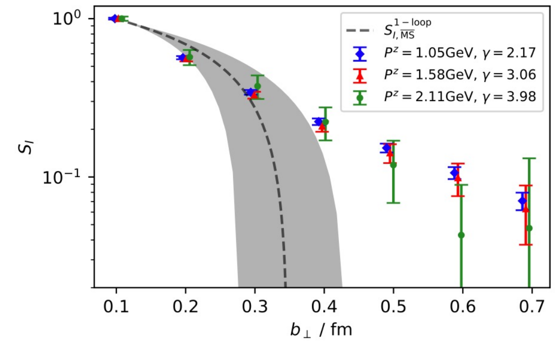
[Shanahan, Wagman, & Zhao (PRD 2021).]





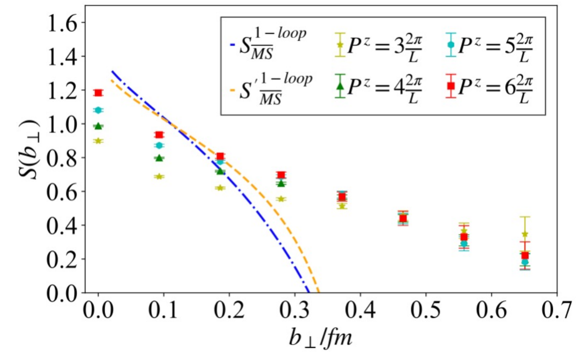
Reduced soft function

[LPC collaboration (PRL 2020).]



Reduced soft function

[Li et al. (PRL 2022).]



Three key ingredients

1. Numerically tractable “Lattice TMDs” ✅
2. Precision lattice calculations ⏳
3. Connection to physical TMDs?

A plethora of TMD definitions…

Modern Collins

$$\tilde{f}_{i/p}(x, \mathbf{b}_T, \mu, \zeta) = \lim_{\epsilon \to 0} Z_{\rm uv}(\mu, \zeta, \epsilon) \lim_{y_B \to -\infty} \frac{\tilde{f}^{0\,(\rm u)}_{i/p}(x, \mathbf{b}_T, \epsilon, y_B, xP^+)}{\sqrt{\tilde{S}^0_{n_A(y \ldots}}}$$

Echevarria, Idilbi, Scimemi

$$\tilde{f}_{i/p}(x, \mathbf{b}_T, \mu, \zeta) = \lim_{\epsilon \to 0} Z^i_{\rm uv}(\mu, \zeta, \epsilon) \frac{\tilde{f}^{0\,(\rm u)}_{i/p}(x, \mathbf{b}_T, \epsilon, \delta^+/(xP^+))}{\sqrt{\tilde{S}^0_{\rm EIS}(b_T, \epsilon, \delta^+ e^{-y_n})}}$$

Chiu, Jain, Neill, Rothstein

$$\tilde{f}_{i/p}(x, \mathbf{b}_T, \mu, \zeta) = \lim_{\substack{\epsilon \to 0 \\ \eta \to 0}} Z^i_{\rm uv}(\mu, \zeta, \epsilon)\, \tilde{f}^{0\,(\rm u)}_{i/p}(x, \mathbf{b}_T, \epsilon, \eta, xP^+) \sqrt{\tilde{S}^0_{\rm CJNR}(b_T, \epsilon, \eta)}$$

Becher & Neubert

$$\lim_{\substack{\epsilon \to 0 \\ \alpha \to 0}} \left[\tilde{f}^{0\rm(u),BN}_{i/p}(x_1, \mathbf{b}_T, \epsilon, \alpha, x_a P^+_A) \tilde{f}^{0\rm(u),BN}_{j/p}(x_2, \mathbf{b}_T, \epsilon, \alpha, x_b P^-_B) \right]$$

$$(b^2 Q^2)^{-\gamma^q_F(\mu, b_T)} \ldots (x_2, \mathbf{b}_T, \mu, \zeta = b_0^2/b_T^2) \Big]$$

Ji, Ma, Yuan

$$\tilde{f}_{i/p}(x_a, \mathbf{b}_T, \mu, x_a \tilde{\zeta}_a; \rho) = \lim_{\epsilon \to 0} Z^i_{\rm uv}(\mu, \rho, \epsilon) \frac{\tilde{f}^{0\,(\rm u)}_{i/p}(x_a, \mathbf{b}_T, \epsilon, v, xP^+)}{\sqrt{\tilde{S}^0_{v\bar{v}}(b_T, \epsilon, \rho)}} + O(v^+, \bar{v}^-) \,.$$

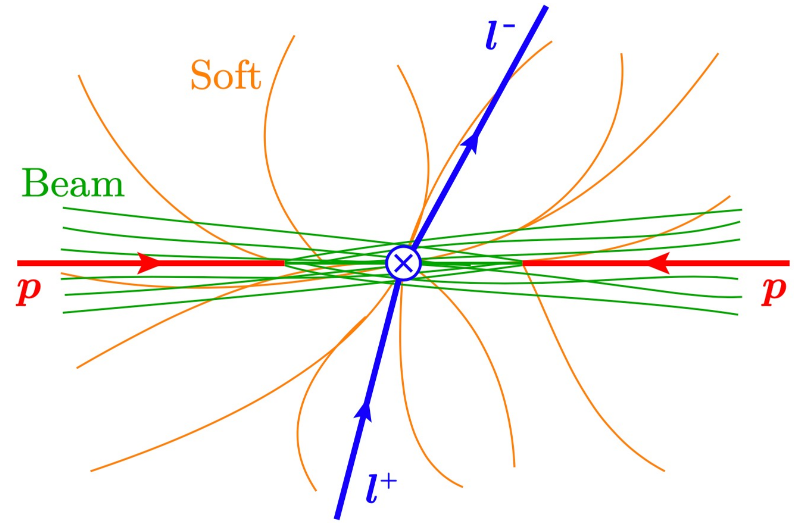
Etc!

Let’s sort this all out!

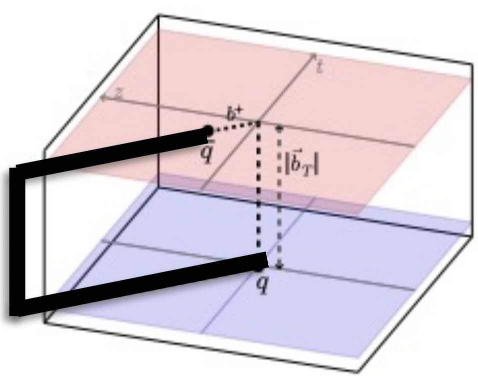
TMD Handbook Ch. 2.

General structure of a TMD

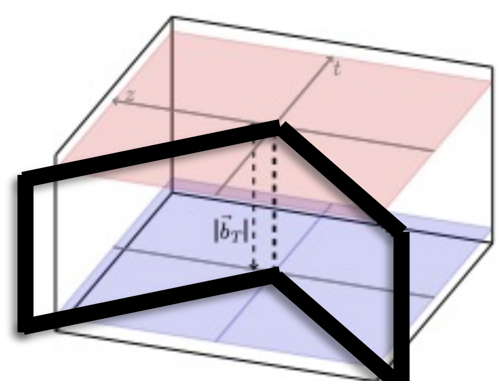
$$f = \lim_{\substack{\text{lightcone,}\\ \text{renormalization}}} Z_{UV} \frac{B^{[\Gamma]}_{q_i/H}}{\sqrt{S^R}}$$



Beam function:

$\langle P|$  $|P\rangle$

Soft factor:

$\langle 0|$  $|0\rangle$

Unifying notation in the literature

Can describe lattice & continuum off-lightcone schemes using <u>the same</u> generic **beam function** & **soft factor**

- Quasi-TMD
- Collins TMD
- JMY scheme
- Etc.

$$f = \lim_{\substack{\text{lightcone,} \\ \text{renormalization}}} Z_{UV} \frac{B^{[\Gamma]}_{q_i/H}(b, P, \epsilon, \eta\nu, \delta)}{\sqrt{S^R(b, \epsilon, \eta\nu, \bar{\eta}\bar{\nu})}}$$

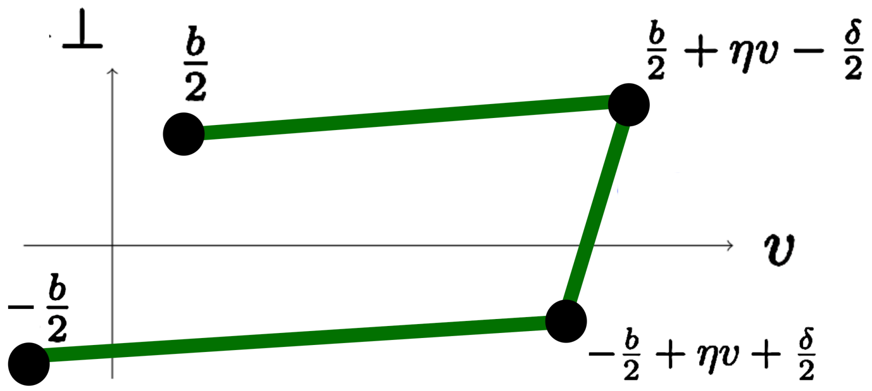
Each scheme is characterized by a distinct set of **arguments** & **limits**

Ebert, **Schindler**, Stewart, and Zhao (JHEP 2022).

Structure of the correlators

$$\textbf{Beam} = \left\langle P \left| \bar{q}_i \frac{\Gamma}{2} W_{\sqsupset}^{F}(b, \eta v, \delta) q_i \right| P \right\rangle$$

$$\textbf{Soft} = \frac{1}{d_R} \langle 0 | \mathrm{Tr}[S_{\gtrless}^{R}(b, \eta v, \overline{\eta v})] | 0 \rangle$$



- **b^μ, ηv^μ, δ^μ:** parametrize Wilson lines
- **Length η:** finite (lattice) or infinite (physical TMD)
- $\delta^\mu = (0,0,0,\tilde{b}^z)$ for quasi
  $= (0,0,0,0)$ for MHENS

Ebert, **Schindler**, Stewart, and Zhao (JHEP 2022).

Neat & tidy charts!

	Collins TMD (continuum)	Quasi-TMD (lattice)
TMD	$\lim_{\epsilon\to 0} Z_{\mathrm{UV}}^{\kappa_i} \lim_{y_B\to-\infty} \frac{\Omega_{i/h}}{\sqrt{S^{\kappa_i}}}$	$\lim_{a\to 0} Z_{\mathrm{UV}}^{\kappa_i} \frac{B_{i/h}}{\sqrt{\tilde{S}^{\kappa_i}}}$
Beam function	$\Omega_{i/h}\left[b, P, \epsilon, -\infty n_B(y_B), b^- n_b\right]$	$\Omega_{i/h}(\tilde{b}, \tilde{P}, a, \tilde{\eta}\hat{z}, \tilde{b}^z\hat{z})$
Soft function	$S^{\kappa_i}\left[b_\perp, \epsilon, -\infty n_A(y_A), -\infty n_B(y_B)\right]$	$S^{\kappa_i}\left[b_\perp, a, -\tilde{\eta}\frac{n_A(y_A)}{\lvert n_A(y_A)\rvert}, -\tilde{\eta}\frac{n_A(y_A)}{\lvert n_A(y_A)\rvert}\right]$
b^μ	$(0, b^-, b_\perp)$	$(0, b_T^x, b_T^y, \tilde{b}^z)$
v^μ	$(-e^{2y_B}, 1, 0_\perp)$	$(0, 0, 0, -1)$
δ^μ	$(0, b^-, 0_\perp)$	$(0, 0, 0, \tilde{b}^z)$
P^μ	$\frac{m_h}{\sqrt{2}}(e^{y_P}, e^{-y_P}, 0_\perp)$	$m_h(\cosh y_{\tilde{P}}, 0, 0, \sinh y_{\tilde{P}})$

Schindler, Stewart, and Zhao (2022).

Outline of today's lecture

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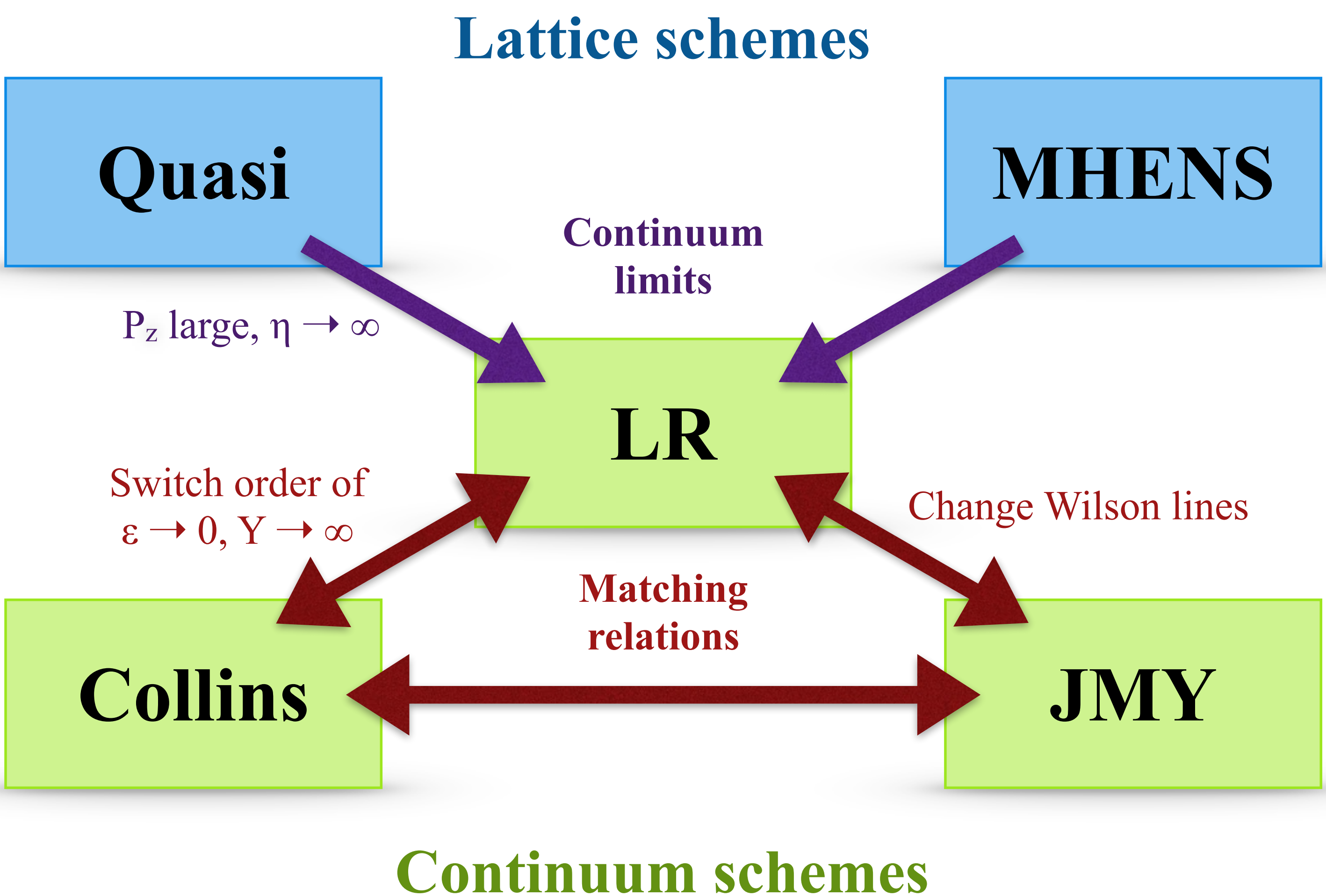
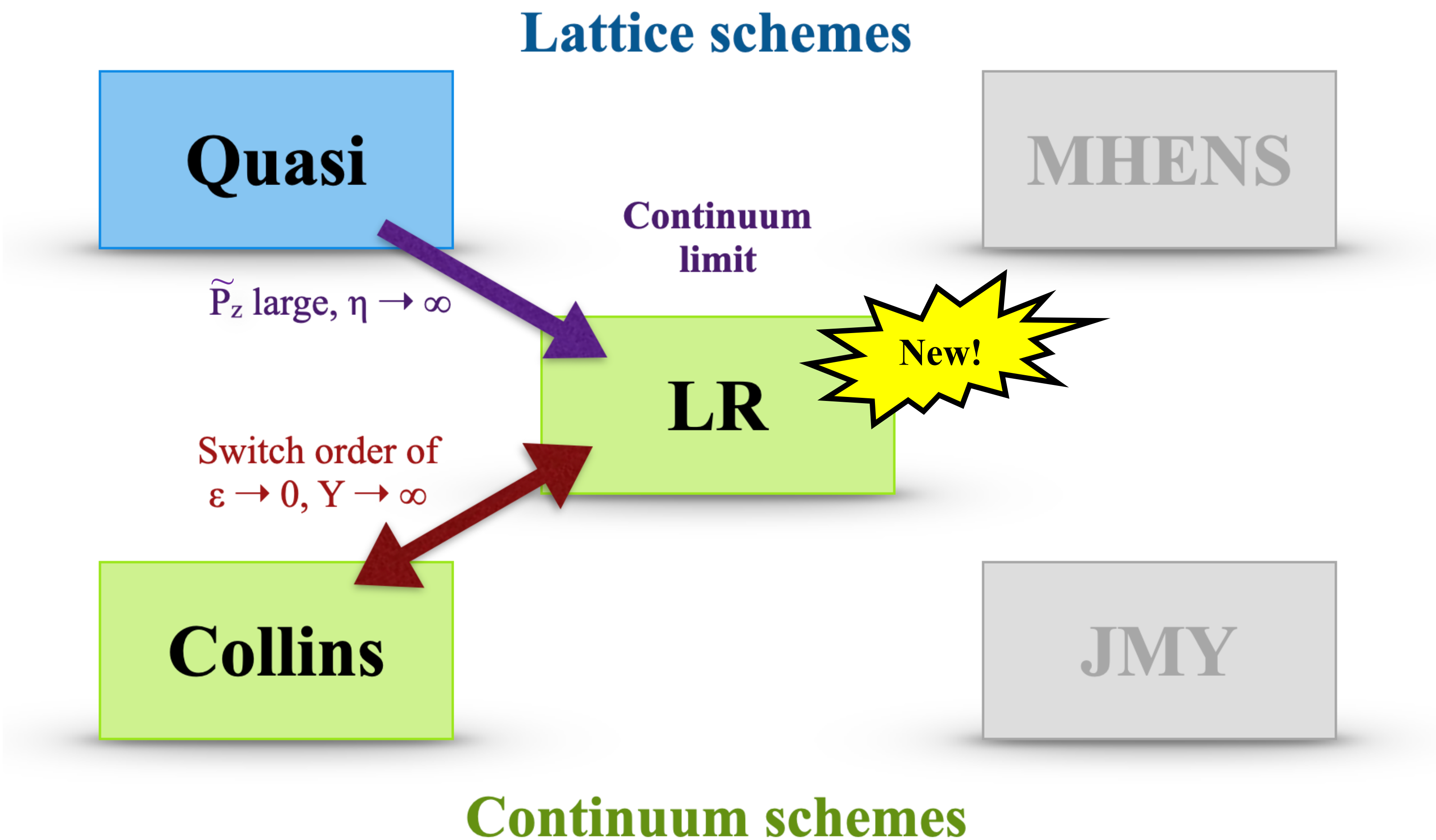
I. Historical overview	🤔	🙂
II. New notation	🤓	🥱
III. Factorization	😎	😱
IV. Outlook	👀	🥳

Unified notation → straightforward to see relationships



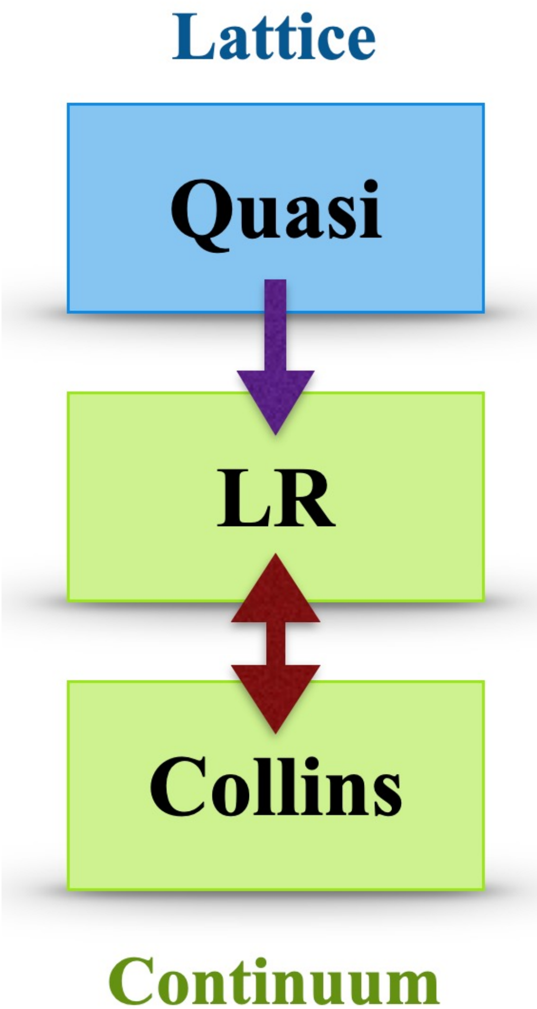
Ebert, **Schindler**, Stewart, and Zhao (JHEP 2022).

Our target



Ebert, **Schindler**, Stewart, and Zhao (JHEP 2022).

Factorization derivation steps



Step 1: same at large rapidity $P^z \gg \Lambda_{QCD}$

- Expand & relate their variables
- Take Wilson line length $|\eta| \to \infty$

Step 2: need a matching coefficient

- Different UV renormalizations
- Nontrivial relationship

Focus on beams: quasi-soft function is chosen to reproduce the Collins soft function

Step 1: Quasi to Large Rapidity

Compare Lorentz invariants formed from beam function arguments b^μ, P^μ, δ^μ, ηv^μ

Use boosts to show quasi = LR as $|\boldsymbol{\eta}| \to \infty$ **&** $\mathbf{P^z >> \Lambda_{QCD}}$

	Quasi	**LR**
b^2	$-b_T^2 - (\tilde{b}^z)^2$	$-b_T^2$
$(\eta v)^2$	$-\tilde{\eta}^2$	$-2\eta^2 e^{2y_B}$
$P \cdot b$	$-m_h \tilde{b}^z \sinh y_{\tilde{P}}$	$\frac{m_h}{\sqrt{2}} b^- e^{y_P}$
$\frac{b \cdot (\eta v)}{\sqrt{\lvert(\eta v)^2 b^2\rvert}}$	$\frac{\tilde{b}^z}{\sqrt{(\tilde{b}^z)^2 + b_T^2}} \,\mathrm{sgn}(\eta)$	$-\frac{b^- e^{y_B}}{\sqrt{2}\, b_T} \,\mathrm{sgn}(\eta)$
$\frac{P \cdot (\eta v)}{\sqrt{P^2 \lvert\eta v\rvert^2}}$	$\sinh y_{\tilde{P}} \,\mathrm{sgn}(\eta)$	$\sinh(y_P - y_B)\,\mathrm{sgn}(\eta)$
$\frac{\delta^2}{b^2}$	$\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$	0
$\frac{b \cdot \delta}{b^2}$	$\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$	0
$\frac{P \cdot \delta}{P \cdot b}$	1	1
$\frac{\delta \cdot (\eta v)}{b \cdot (\eta v)}$	1	1
P^2	m_h^2	m_h^2

Step 1: Quasi to Large Rapidity

Examine all 10 Lorentz invariants:

Need $\tilde{\eta} = \sqrt{2}\, e^{y_B} \eta$

⋮

Need $y_P - y_B = y_{\tilde{P}}$

⋮

As $y_{\tilde{P}} \to -\infty$, $b_T \gg \tilde{b}_z$

Quasi = LR after large rapidity expansion ✅

	Quasi	LR
b^2	$-b_T^2 - (\tilde{b}^z)^2$	$-b_T^2$
$(\eta v)^2$	$-\tilde{\eta}^2$	$-2\eta^2 e^{2y_B}$
$P \cdot b$	$-m_h \tilde{b}^z \sinh y_{\tilde{P}}$	$\frac{m_h}{\sqrt{2}} b^- e^{y_P}$
$\frac{b \cdot (\eta v)}{\sqrt{\lvert(\eta v)^2 b^2\rvert}}$	$\frac{\tilde{b}^z}{\sqrt{(\tilde{b}^z)^2 + b_T^2}} \operatorname{sgn}(\eta)$	$-\frac{b^- e^{y_B}}{\sqrt{2}\, b_T} \operatorname{sgn}(\eta)$
$\frac{P \cdot (\eta v)}{\sqrt{P^2 \lvert\eta v\rvert^2}}$	$\sinh y_{\tilde{P}} \operatorname{sgn}(\eta)$	$\sinh(y_P - y_B) \operatorname{sgn}(\eta)$
$\frac{\delta^2}{b^2}$	$\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$	0
$\frac{b \cdot \delta}{b^2}$	$\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$	0
$\frac{P \cdot \delta}{P \cdot b}$	1	1
$\frac{\delta \cdot (\eta v)}{b \cdot (\eta v)}$	1	1
P^2	m_h^2	m_h^2

Step 2: Large Rapidity to Collins

	TMD	Beam function	Soft function
Collins	$\lim\limits_{\epsilon\to 0}\lim\limits_{y_B\to-\infty} Z_{UV}^{R}\frac{\Omega_{i/h}}{\sqrt{S^R}}$	$\Omega_{q/h}^{[\gamma^+]}\left[b, P, \epsilon, -\infty n_B(y_B), b^- n_b\right]$	$S^R\left[b_\perp, \epsilon, -\infty n_A(y_A), -\infty n_B(y_B)\right]$
LR	$\lim\limits_{-y_B\gg 1}\lim\limits_{\epsilon\to 0} Z_{UV}^{R}\frac{\Omega_{i/h}}{\sqrt{S^R}}$	$\Omega_{q/h}^{[\gamma^+]}\left[b, P, \epsilon, -\infty n_B(y_B), b^- n_b\right]$	$S^R\left[b_\perp, \epsilon, -\infty n_A(y_A), -\infty n_B(y_B)\right]$

Fundamental principle of EFT (here, LaMET):

- Flipping an order of UV limits does not affect IR physics
- However, it can induce a perturbative matching coefficient

$$f_{LR} = C_i\left(x\tilde{P}^z, \mu\right) f_{Collins}$$

Steps 1 + 2 → Factorization

Quasi-TMD (lattice) | Matching | RGE for ζ | **Collins TMD (continuum)**

$$\tilde{f}^{[s]}_{i/H}\left(x, \vec{b}_T, \mu, \tilde{\zeta}, x\tilde{P}^z\right) = C_i\left(x\tilde{P}^z, \mu\right) \exp\left[\frac{1}{2}\gamma_\zeta^i(\mu, b_T) \ln\frac{\tilde{\zeta}}{\zeta}\right] f^{[s]}_{i/H}\left(x, \vec{b}_T, \mu, \zeta\right)$$

$$\tilde{\zeta} = \left(2x\tilde{P}^z\right)^2 e^{2(y_B - y_n)}$$

Power corrections

$$\times\left\{1 + \mathcal{O}\left[\frac{1}{\left(x\tilde{P}^z b_T\right)^2}, \frac{\Lambda_{QCD}^2}{\left(x\tilde{P}^z\right)^2}\right]\right\}$$

Note that this formula connects physical continuum TMDs to the renormalized ***continuum limit*** of lattice calculations.

Ebert, **Schindler**, Stewart, and Zhao (JHEP 2022).

Matching coefficient?

$$\tilde{f}_{i/H}^{[s]}\left(x, \vec{b}_T, \mu, \tilde{\zeta}, x\tilde{P}^z\right) = \boldsymbol{C_i\left(x\tilde{P}^z, \mu\right)} \exp\left[\frac{1}{2}\gamma_\zeta^i(\mu, b_T) \ln\frac{\tilde{\zeta}}{\zeta}\right] f_{i/H}^{[s]}\left(x, \vec{b}_T, \mu, \zeta\right)$$

NLO:

$$C_i\left(\mu, x\tilde{P}^z\right) = 1 + \frac{\alpha_s C_R}{4\pi}\left[-\ln^2\frac{(2xP^z)^2}{\mu^2} + \frac{2\ln(2xP^z)^2}{\mu^2} - 4 + \frac{\pi^2}{6}\right] + O(\alpha_s^2)$$

N*ⁿ*LL:

$$C_i\left(x\tilde{P}^z, \mu\right) = C_i[\alpha_s(\mu)] \exp\left[\int_{\alpha_s(\mu)}^{\alpha_s(2x\tilde{P}^z)} \frac{d\alpha}{\beta[\alpha]} \int_{\alpha}^{\alpha_s(\mu)} \frac{d\alpha'}{\beta[\alpha']} \left(2\boldsymbol{\Gamma_{cusp}^i[\alpha']} + \boldsymbol{\gamma_C^i[\alpha]}\right)\right]$$

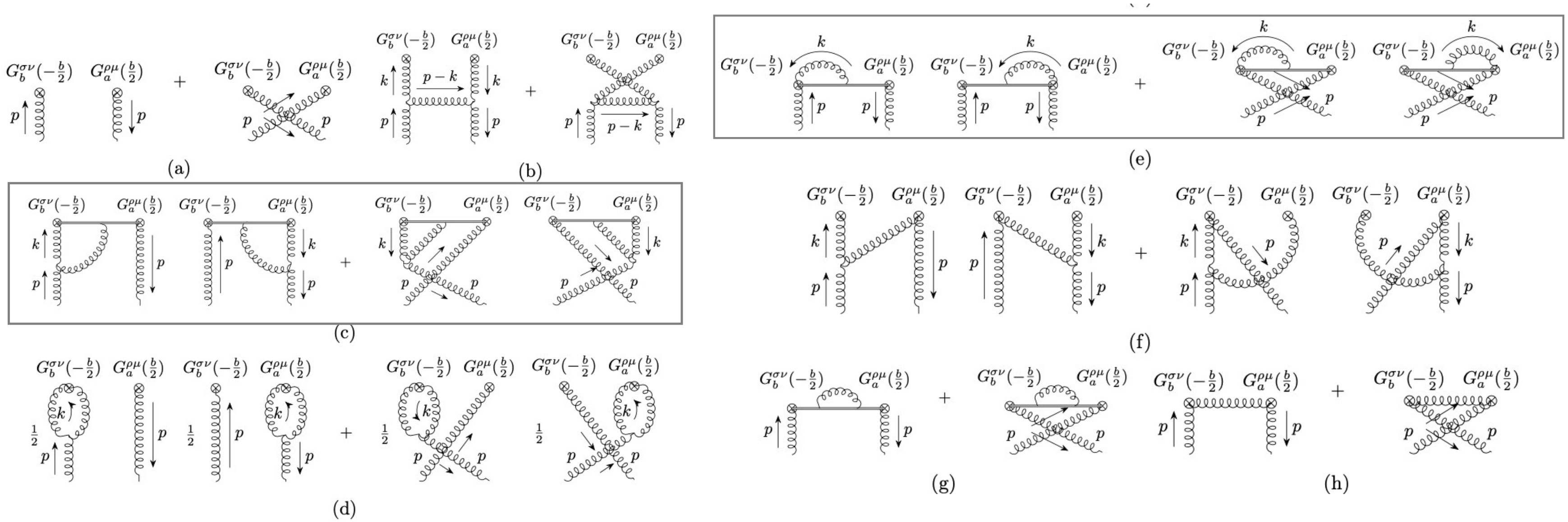
Etc.

Focus on general features, not calculations…

Ebert, **Schindler**, Stewart, and Zhao (JHEP 2022). **Schindler**, Stewart, and Zhao (2022).

Gluon matching coefficient at NLO

Focus on general features, not calculations…



(Key simplification: only rapidity-divergent pieces can contribute.)

NLO: Casimir scaling for quarks and gluons

$$C_i\left(\mu, x\tilde{P}^z\right) = 1 + \frac{\alpha_s C_R}{4\pi}\left[-\ln^2\frac{(2xP^z)^2}{\mu^2} + \frac{2\ln(2xP^z)^2}{\mu^2} - 4 + \frac{\pi^2}{6}\right] + O(\alpha_s^2)$$

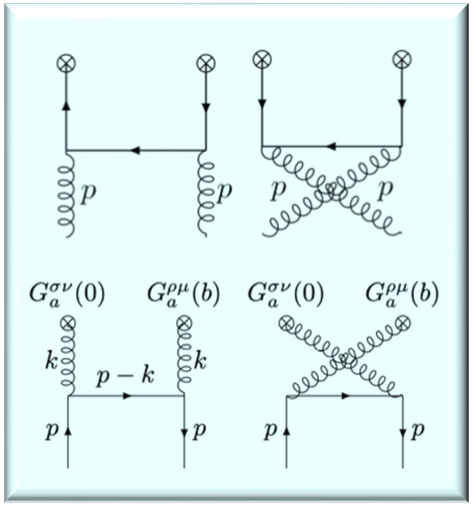
Schindler, Stewart, and Zhao (2022).

Matching coefficient

C_i is independent of spin and quark flavor

Two coefficients C_q & C_g: no quark-gluon mixing

Hadron polarization		Quark polarization		
		U	L	T
	U	f_1 unpolarized		$h_1^\perp$ Boer-Mulders
	L		g_{1L} helicity	$h_{1L}^\perp$ worm-gear
	T	$f_{1T}^\perp$ Sivers	g_{1T} worm-gear	h_1, $h_{1T}^\perp$ transversity, pretzelosity



Ebert, **Schindler**, Stewart, and Zhao (JHEP 2020, JHEP 2022).

TMD ratios

Can extract TMD spin/flavor/hadron ratios from lattice beam functions:

$$\lim_{\tilde{\eta}\to\infty} \frac{f_{q_i/h}^{[\tilde{\Gamma}_1]}}{f_{q_j/h'}^{[\tilde{\Gamma}_2]}} = \lim_{\tilde{\eta}\to\infty} \frac{\tilde{B}_{q_i/h}^{[\tilde{\Gamma}_1]}}{\tilde{B}_{q_j/h'}^{[\tilde{\Gamma}_2]}}$$

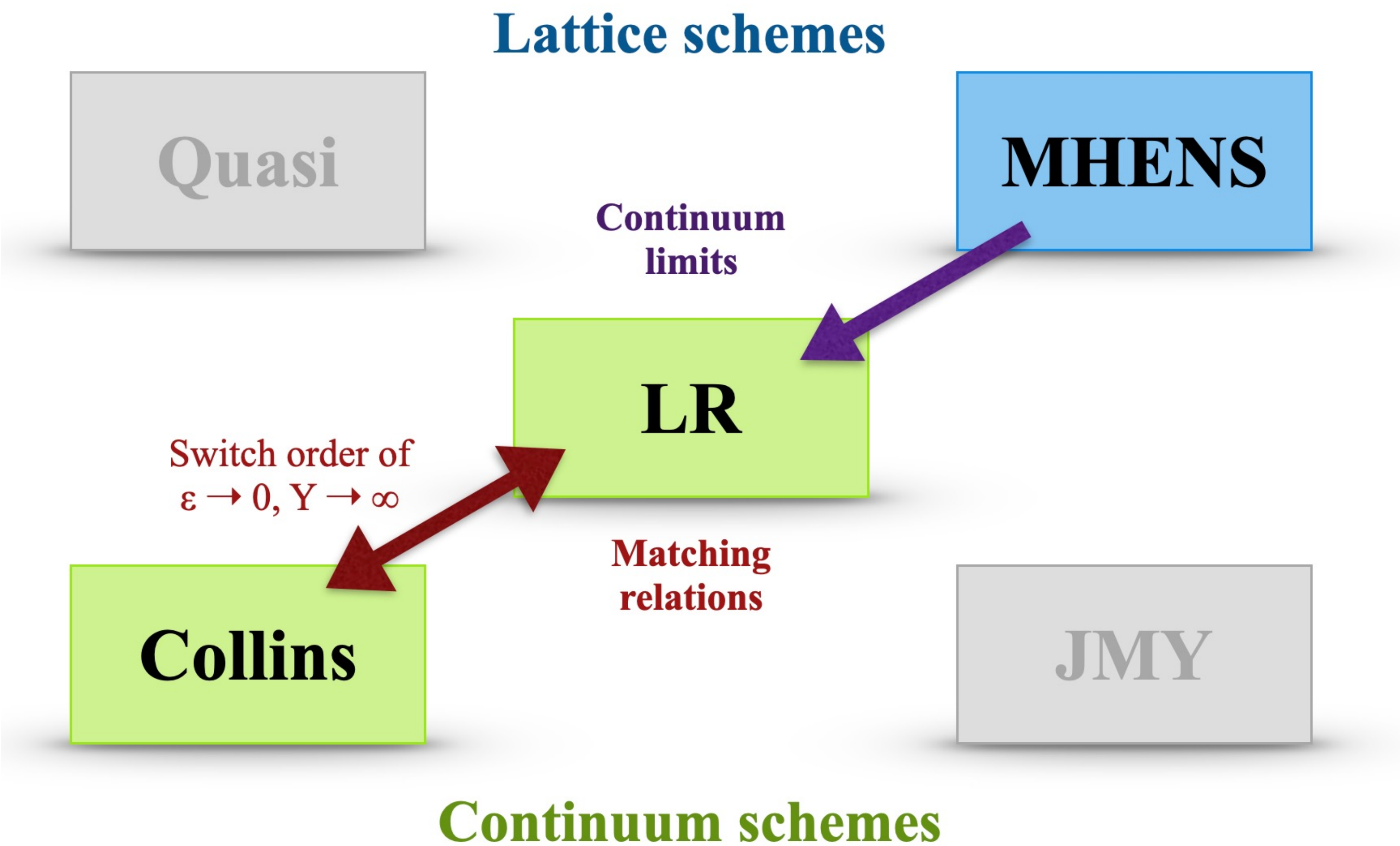
Can see from factorization formulas:

$$C_i \exp\left[\frac{1}{2}\gamma_\zeta^i \ln\frac{\tilde{\zeta}}{\zeta}\right] f_{q_i/H}^{[\Gamma]} = \tilde{f}_{q_i/H}^{[\Gamma]} = \lim Z_{UV} \frac{\tilde{B}_{q_i/H}^{[\Gamma]}}{\sqrt{S^R}}$$

Lattice-to-continuum TMD factorization

Factorization of a lattice TMD into matrix elements

MHENS-to-Collins factorization



Ebert, **Schindler**, Stewart, and Zhao (JHEP 2022).

MHENS-to-Collins at $P \cdot b = 0$

This case was the focus of the MHENS authors. Equivalent soft function, renormalization, etc. as quasi-TMDs:

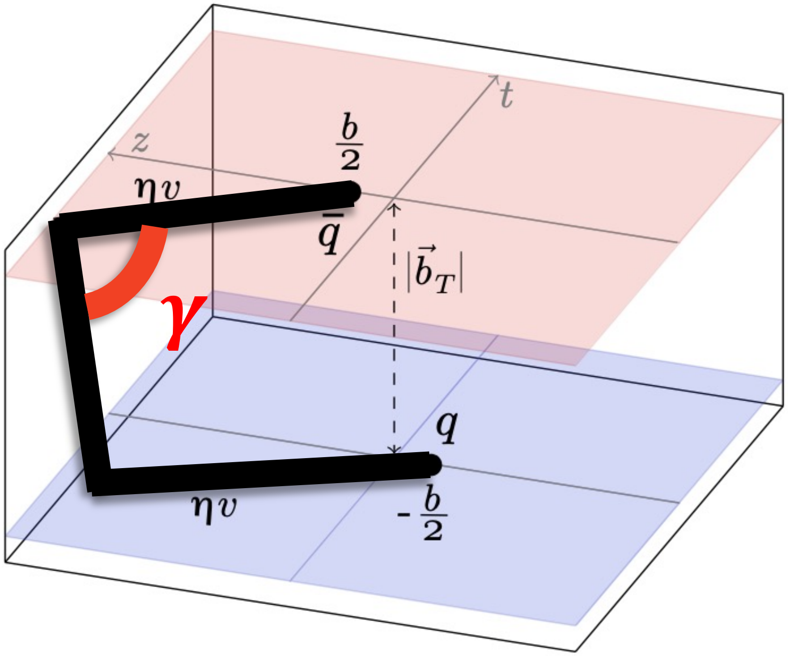
$$\int \mathrm{d}x \; \tilde{f}^{[\Gamma]}_{q_i/h}(x, \vec{b}_T, \mu, \tilde{\zeta}, x\tilde{P}^z, \tilde{\eta}) = \; f^{[\Gamma]\mathrm{MHENS}}_{q_i/h}(b^z = 0, \vec{b}_T, \mu, \tilde{P}^z, y_n - y_B, \tilde{\eta})$$

So, factorization is straightforward, involves a convolution:

$$\lim_{\tilde{\eta}\to\infty} \tilde{f}^{[\Gamma]\mathrm{MHENS}}_{q_i/h}(b^z = 0, \vec{b}_T, \mu, \tilde{P}, y_n - y_B, \tilde{\eta}) = \int \mathrm{d}x \; C_q(x\tilde{P}^z, \mu) \; f^{[\Gamma]}_{q_i/h}(x, \vec{b}_T, \mu, \zeta)$$

Thus, our factorization derivation implies that all MHENS scheme calculations carried out so far have a rigorous connection to physical TMDs.

MHENS at $P \cdot b \neq 0$ (x dependence)



b^z-dependent **Wilson line length**:

$$L_{\text{staple}}^{\text{MHENS}} = 2|\tilde{\eta}v| + |b|$$

Nontrivial **cusp angles**, even as $\eta \to \infty$:

$$\cosh[\gamma(v,b)] = \pm\frac{v \cdot b}{|v||b|}$$

Length of a four-vector: $|X| = \sqrt{|X^2|}$

Complications:

- ➢ Renormalization & soft function would be b^z-dependent
- ➢ These won't cancel out in ratios at finite η

Summary of results

1. New unified notation

2. New scheme (LR)

3. **Continuum-to-lattice factorization**

4. Matching coefficient: convenient!

Take-home messages

When constructing a lattice observable, it is helpful to consider the full phase space of options.

Balancing analytic & numerical challenges…

- Computational cost
- Relationship with physical observable
- Proper definition (renormalization, soft function, finiteness)

There is much to pursue on the lattice!

Proposal to rebrand quasi-TMDs

Lattice TMDs: MHENS and LADIEZ

LaMET

Approach

Developed (in part) by

Iain,

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