Axial Gauge and Wilson Lines of Infinite Extent

Xiaojun Yao
MIT

Bruno Scheihing-Hitschfeld, XY, 2205.04477

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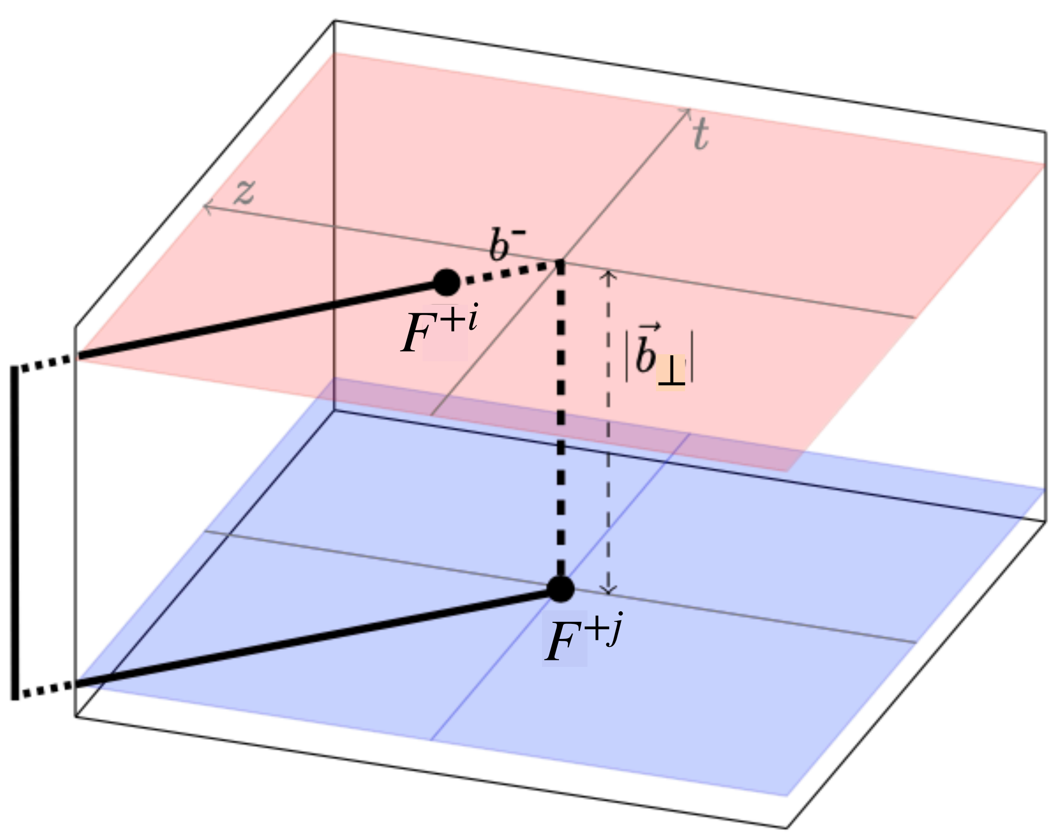
Two Gluon TMDs

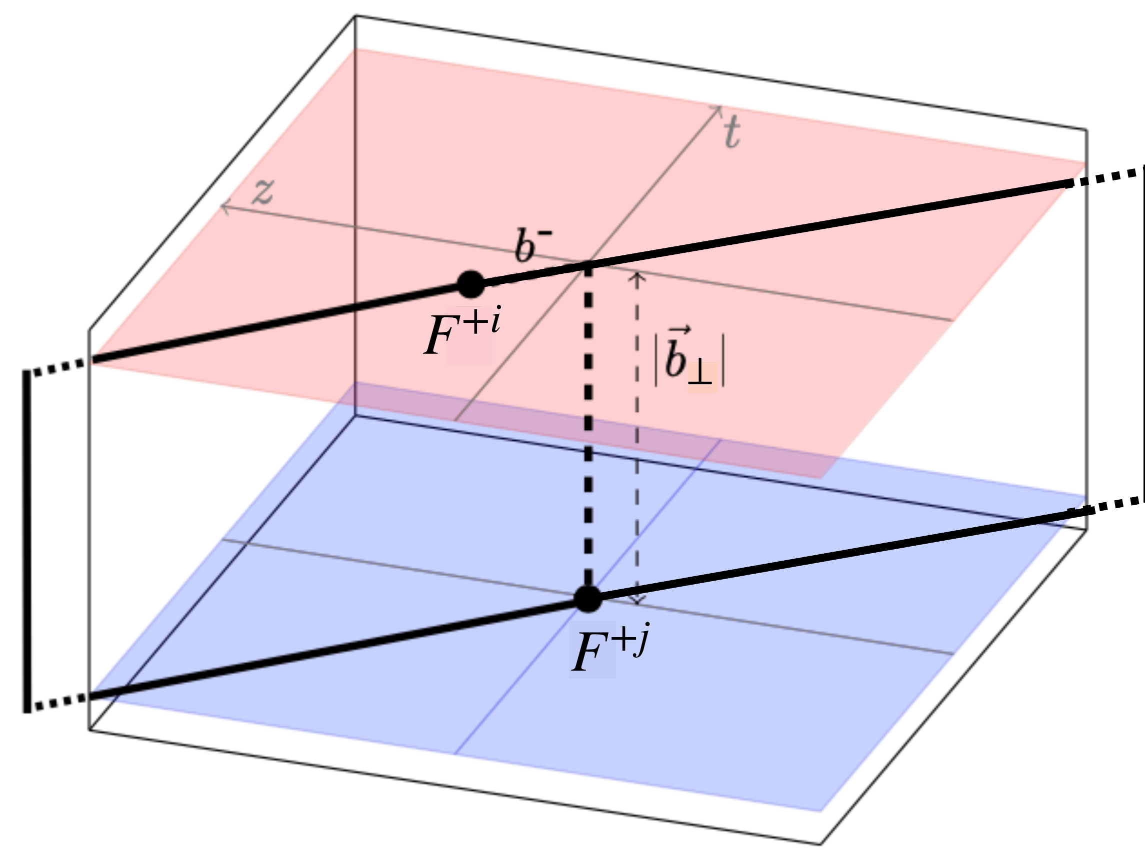
- **Weizsacker-Williams TMD (in standard TMD factorization)** See TMD handbook

$$\frac{1}{xP^+}\int\frac{db^-db_\perp^2}{2(2\pi)^3}e^{-ixb^-P^+-ib_\perp\cdot k_\perp}T_F\langle p(P,S)\big|F^{a+i}(b^-,b_\perp)\mathcal{W}^{ab}_{[(b^-,b_\perp),(-\infty^-,b_\perp)]}$$
$$\mathcal{W}^{bc}_{[(-\infty^-,b_\perp),(-\infty^-,0_\perp)]}\mathcal{W}^{cd}_{[(-\infty^-,0_\perp),(0^-,0_\perp)]}F^{d+j}(0^-,0_\perp)\big|p(P,S)\rangle$$

- **Dipole TMD (in small-x physics)** D. Kharzeev, Y.V. Kovchegov, K. Tuchin, hep-ph/0307037
F. Dominguez, B.-W. Xiao, F. Yuan, 1009.2141

$$\frac{1}{xP^+}\int\frac{db^-db_\perp^2}{2(2\pi)^3}e^{-ixb^-P^+-ib_\perp\cdot k_\perp}\langle p(P,S)\big|\mathrm{Tr}_c\big[U_{[(-\infty^-,0_\perp),(-\infty^-,b_\perp)]}U_{[(-\infty^-,b_\perp),(b^-,b_\perp)]}F^{+i}(b^-,b_\perp)$$
$$U_{[(b^-,b_\perp),(+\infty^-,b_\perp)]}U_{[(+\infty^-,b_\perp),(+\infty^-,0_\perp)]}U_{[(+\infty^-,0_\perp),(0^-,0_\perp)]}F^{+j}(0^-,0_\perp)U_{[(0^-,0_\perp),(-\infty^-,0_\perp)]}\big]\big|p(P,S)\rangle$$





Two Gluon TMDs

- **Different $k_\perp$ dependence**

	Small $k_\perp$	Large $k_\perp$
WW TMD	$\sim \dfrac{\ln k_\perp}{k_\perp}$	$\sim \dfrac{1}{k_\perp^2}$
Dipole TMD	~ 0	$\sim \dfrac{1}{k_\perp^2}$

- **Expect them to have different values after integrating $k_\perp$**

WW TMD
$$\frac{1}{xP^+}\int \frac{db^-}{2(2\pi)} e^{-ixb^-P^+} T_F \langle p(P) | F^{a+i}(b^-)\mathcal{W}^{ab}_{[b^-,0^-]} F^{b+j}(0^-) | p(P)\rangle$$

Dipole TMD
$$\frac{1}{xP^+}\int \frac{db^-}{2(2\pi)} e^{-ixb^-P^+} \langle p(P) | \mathrm{Tr}_c\big[U_{[-\infty^-,b^-]} F^{+i}(b^-) U_{[b^-,0^-]} F^{+j}(0^-) U_{[0^-,-\infty^-]}\big] | p(P)\rangle$$

$$U_{[-\infty^-,0^-]} F^a_{\mu\nu}(0) T^a_F U_{[0^-,-\infty^-]} = T^a_F \mathcal{W}^{ab}_{[-\infty^-,0^-]} F^b_{\mu\nu}(0)$$

Not true quantum mechanically

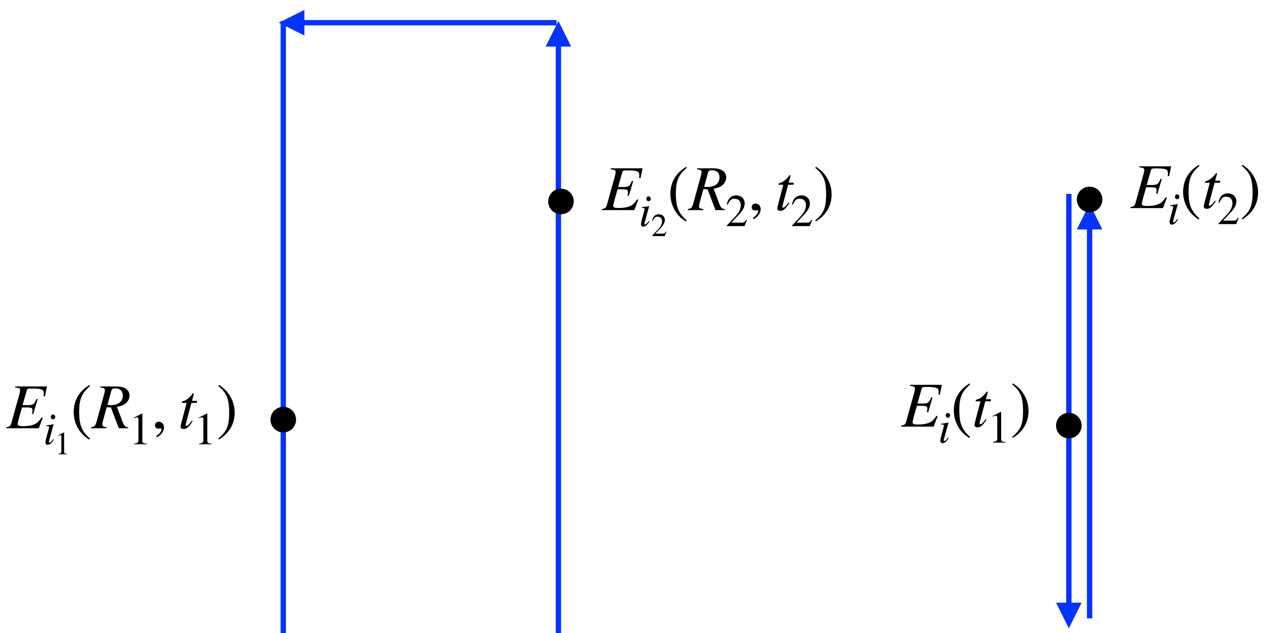
- **However, they look identical in lightcone axial gauge $n \cdot A = A^+ = 0$**

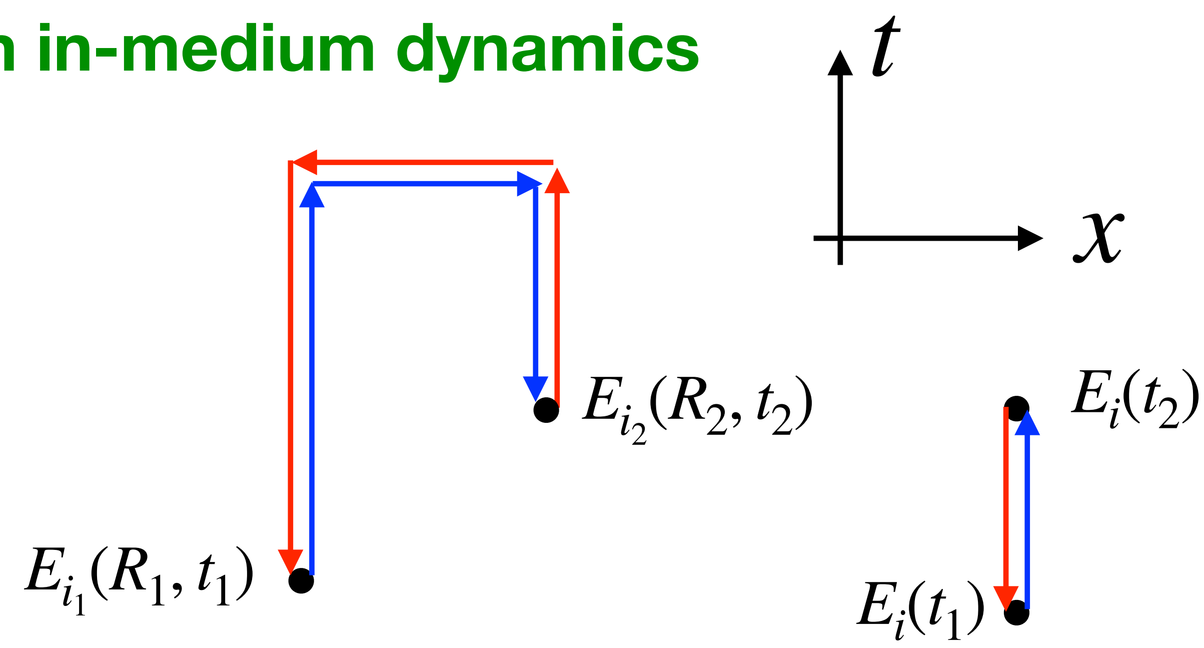
$$\frac{1}{xP^+}\int \frac{d\,b^-}{2(2\pi)} e^{-ixb^-P^+} \langle p(P) | \mathrm{Tr}_c\big[F^{+i}(b^-) F^{+j}(0^-)\big] | p(P)\rangle$$

What's wrong?

Two Chromoelectric Field Correlators

- **Correlators for heavy quark and quarkonium in-medium dynamics**





$$\langle O \rangle_T = \mathrm{Tr}(O \rho_T)$$

Single heavy quark

$$g_E^{\mathrm{Q}}(t) = g^2 \left\langle \mathrm{Tr}_c \left(U_{[-\infty,t]} E_i(t) U_{[t,0]} E_i(0) U_{[0,-\infty]} \right) \right\rangle_T$$

J.Casalderrey-Solana, D.Teaney, hep-ph/0605199

Heavy quark antiquark pair

$$g_E^{\mathrm{Q}\bar{\mathrm{Q}}}(t) = g^2 T_F \left\langle \left(E_i^a(t) \mathcal{W}_{[t,0]}^{ab} E_i^b(0) \right) \right\rangle_T$$

Thomas Mehen, XY, 2009.02408

Color interactions in
both initial **and** final states
since HQ carries color

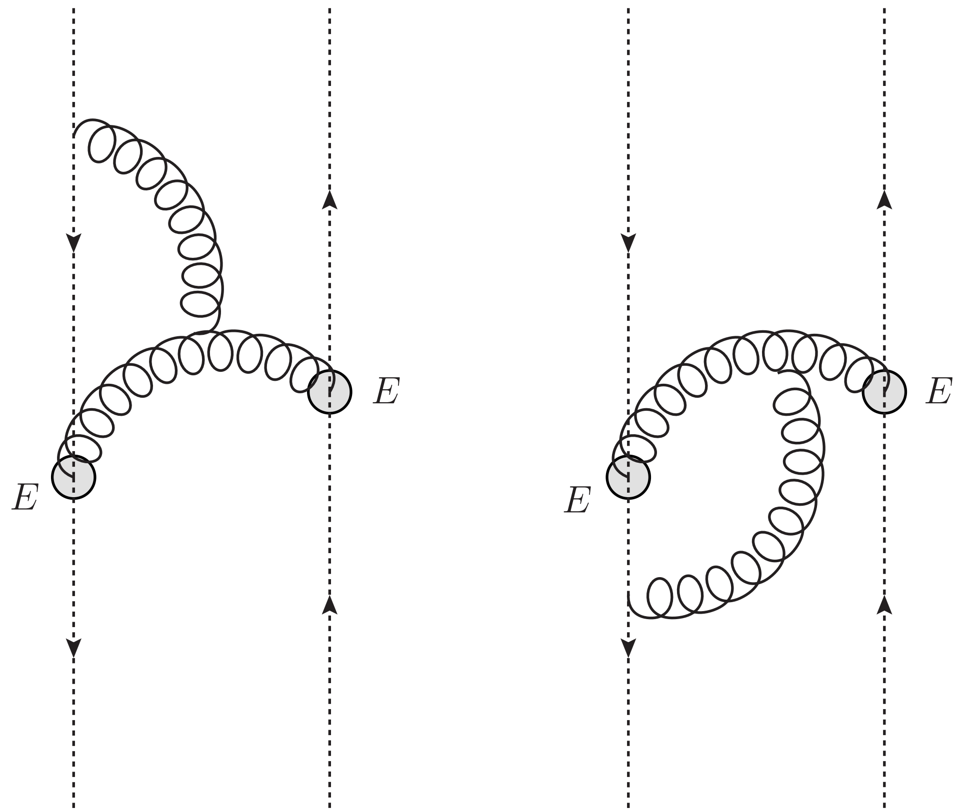
Color interactions in
either initial **or** final state
since quarkonium colorless

Two Chromoelectric Field Correlators

- **At NLO they have different values**

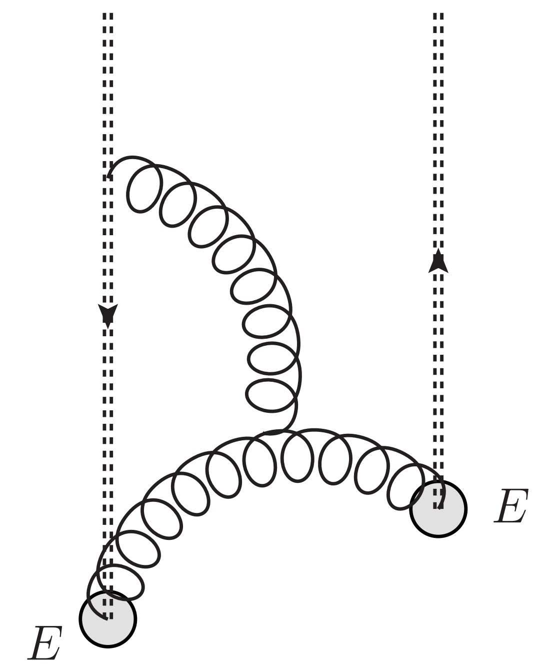
$$\int_{-\infty}^{+\infty} dt\, e^{ip_0 t}\left(g_E^{\mathrm{Q}\bar{\mathrm{Q}}}(t) - g_E^{\mathrm{Q}}(t)\right) = \frac{g^4 N_c (N_c^2 - 1) T_F p_0^3}{(2\pi)^3}\pi^2$$

g_E^Q



Y.Burnier, M.Laine, J.Langelage, L.Mether, 1006.0867

$g_E^{Q\bar{Q}}$



T.Binder, K.Mukaida, B.Scheihing-Hitschfeld, XY, 2107.03945

- **However, they look identical in temporal axial gauge** $A_0 = 0$

$$g^2 T_F \left\langle \mathrm{Tr}_c\left[E_i(t)E_i(0)\right]\right\rangle_T$$

What's wrong?

Axial Gauge Puzzle

- **Axial gauge in Faddeev-Popov path integral**

Gauge condition $G^a_A[A] = n^\mu A^a_\mu(x)$

$$\int \mathcal{D}\omega\, e^{-\frac{i}{2\xi}\int d^4x\, \omega^a\omega^a} \int \mathcal{D}A \det\left(\frac{\delta G^a(x)}{\delta\theta^b(y)}\right) \prod_{x,a} \delta\left(G^a(x) - \omega^a(x)\right) e^{iS_{\mathrm{YM}}[A^a]}$$

- **Action with the gauge fixing part**

$$\frac{i}{2}\int d^4k\, A^{\mu a}(-k)\Big(-g_{\mu\nu}(k^2 + i\varepsilon) + k_\mu k_\nu - \frac{1}{\xi} n_\mu n_\nu\Big) A^{\nu a}(k)$$

Boundary condition

- **Obtain propagator in axial gauge**

$$[D_T(k)]^{ab}_{\mu\nu} = \frac{i\delta^{ab}}{k^2 + i\varepsilon}\left[-g_{\mu\nu} + \frac{n\cdot k\,(k_\mu n_\nu + n_\mu k_\nu) - n^2 k_\mu k_\nu}{(n\cdot k)^2 + i\varepsilon}\right]$$

Using this propagator reproduces Feynman gauge result of $g_E^{Q\bar{Q}}$

Origin of Axial Gauge Puzzle

- **Study a more general gauge choice**

$$G_M^a[A] = \frac{1}{\lambda} n^\mu A_\mu^a(x) + \partial^\mu A_\mu^a(x)$$

Feynman gauge: $\lambda \to \infty,\ \xi = 1$ Axial gauge: $\lambda \to 0,\ \text{any}\ \xi$

For $\xi = 1$

$$[D_T(k)]_{\mu\nu}^{ab} = \frac{i\delta^{ab}}{k^2 + i\varepsilon}\left[-g_{\mu\nu} + \frac{k_\mu n_\nu \left(n\cdot k - i\lambda k^2\right) + n_\mu k_\nu \left(n\cdot k + i\lambda k^2\right) - n^2 k_\mu k_\nu}{(n\cdot k)^2 + \lambda^2 (k^2)^2 + (1 + 2\lambda^2 k^2) i\varepsilon} \right]$$

- **Issue arises in the order of taking limits**

$$\int \frac{d^4k}{(2\pi)^4} \frac{\eta}{(n\cdot k)^2 + \eta^2} \left[D_T(k)\right]_{\nu\mu} n^\mu N(p,k)$$

$$U_{[(+\infty)n^\mu, 0]} = \mathrm{P}\exp\left(ig \int_0^{+\infty} ds\, e^{-\eta s} n^\mu A_\mu(s n^\mu) \right)$$

If $\lambda \to 0$ is taken first, vanishing result

If $\eta \to 0$ is taken first, non-vanishing result

Axial gauge puzzle associated w/ Wilson lines of infinite extent

Nonperturbative Perspective: Abelian

- **Consider a gauge transformation from Feynman to axial in Abelian case:**

$$G_F(x) = \partial_\mu A^\mu(x) \to \partial_\mu A^\mu(x) - \partial^2\theta(x) = n_\mu A^\mu(x) = G_A(x)$$

In momentum space $$\theta(k) = \frac{1}{k^2}\big(n_\mu A^\mu(k) + ik_\mu A^\mu(k)\big)$$

- **Gauge field transforms as**

$$A^\mu(k) \to M^\mu{}_\nu A^\nu(k)$$

$$M^\mu{}_\nu = g^\mu{}_\nu + \frac{ik^\mu}{k^2}\big(n_\nu + ik_\nu\big)$$

- **Transformation matrix has zero eigenvalue**

$$M^\mu{}_\nu k^\nu = i\frac{n\cdot k}{k^2}k^\mu$$ **Zero eigenvalue for $n\cdot k = 0$, Jacobian = 0**

Obstruction at infinite “time” $$A(\bar{n}\cdot x) = \int d(n\cdot k)\, e^{i(\bar{n}\cdot x)(n\cdot k)} A(n\cdot k)$$

Nonperturbative Perspective: Non-Abelian

- **In non-Abelian case:**

$$A'_\mu(x) = V(x)A_\mu(x)V^{-1}(x) - \frac{i}{g}\big(\partial_\mu V(x)\big)V^{-1}(x) \qquad V(x) = e^{i\theta^a(x)T_F^a}$$

For $V(x)$ properly defined at $\bar{n}\cdot x \to \infty$ $\qquad \lim_{\bar{n}\cdot x\to\infty} n^\mu \partial_\mu \theta^a(x) = 0$

At $\bar{n}\cdot x \to \infty$ $\qquad n\cdot A'(x) = V(x) n\cdot A(x) V^{-1}(x)$

Thus $\mathrm{Tr}\big[\big(n^\mu A_\mu(\bar{n}\cdot x = \infty)\big)^2\big]$ cannot be changed by gauge transformation

Cannot smoothly go from a gauge with $n\cdot A(\bar{n}\cdot x \to \infty) \neq 0$ to axial gauge $n\cdot A = 0$

Interpretations

- Gauge transformation towards axial gauge breaks down at $\bar{n} \cdot x \to \infty$

- Not always a problem in calculations

$$\frac{\int \mathcal{D}A\, e^{iS[A]}\, O[A]}{\int \mathcal{D}A\, e^{iS[A]}}$$

  - If operator $O[A]$ contains no fields at $\bar{n} \cdot x \to \infty$ —> **canceled** ✅

  - If operator $O[A]$ contains fields at $\bar{n} \cdot x \to \infty$ —> **problem!** ❌

- Axial gauge works fine for integrated WW gluon TMD & quarkonium EE correlator because only finite-extended Wilson lines involved

- For unintegrated TMDs and momentum dependent EE correlators, axial gauge is also fine in practical calculations because transverse/spatial Wilson lines provide proper regularization for the Wilson lines in the time direction, as in the case of A.V. Belitsky, X. Ji, F. Yuan, hep-ph/0208038

Conclusions

- Axial gauge puzzle:
  - Two integrated gluon TMDs and two chromoelectric field correlators look identical in axial gauge, but perturbative calculations show they are different
  - Origin of the issue: break down of gauge transformation towards axial gauge at $\bar{n} \cdot x \to \infty$
  - Not always give incorrect results
- How significant is the difference in the Wilson line configurations nonperturbatively? Important for phenomenology

Backup: Sum Interactions at Leading Order in v

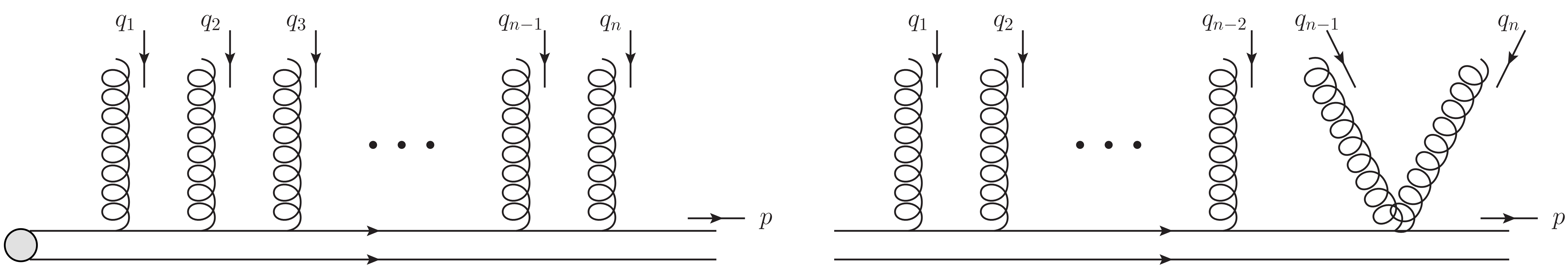
$$\int \mathrm{d}^3 r\, \mathrm{Tr}\left(\mathrm{O}^\dagger(\boldsymbol{R},\boldsymbol{r},t)\left(iD_0 + \boxed{\frac{\boldsymbol{D}_{\boldsymbol{R}}^2}{4M}} + \frac{\nabla_{\boldsymbol{r}}^2}{M} - V_o(\boldsymbol{r}) + \cdots \right)\mathrm{O}(\boldsymbol{R},\boldsymbol{r},t)\right)$$

$$\mathrm{O}(\boldsymbol{R},\boldsymbol{r},t) = \mathcal{W}_{[(\boldsymbol{R},t),(\boldsymbol{R},t_0)]}\widetilde{\mathrm{O}}(\boldsymbol{R},\boldsymbol{r},t)$$

$$\mathcal{W}_{[(\boldsymbol{R},t_f),(\boldsymbol{R},t_i)]} = \mathcal{P}\exp\left(ig\int_{t_i}^{t_f} ds\mathcal{A}_0(\boldsymbol{R},s)\right)$$

C.M. kinetic term same order as D_0 for Coulomb modes

$$p_c^\mu \sim A_c^\mu \sim M(v^2, v, v, v)$$



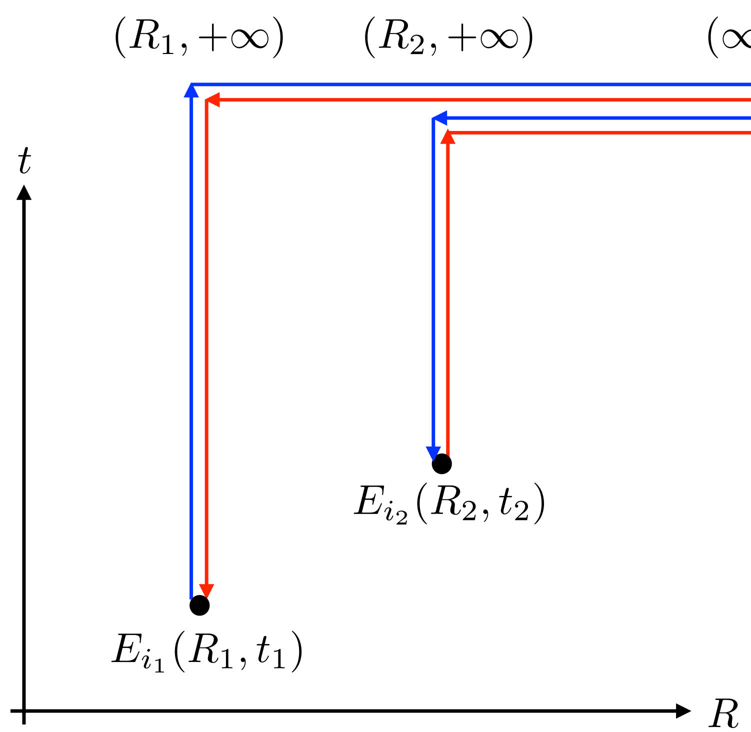
Lead to Wilson lines at infinite time

T.Mehen, XY, 2009.02408

Backup: Chromoelectric Field Correlator

$$[g_E^{++}]^{>}_{ji}(y,x) \equiv \Big\langle \big[E_j(y)\mathcal{W}_{[(y^0,\boldsymbol{y}),(+\infty,\boldsymbol{y})]}\mathcal{W}_{[(+\infty,\boldsymbol{y}),(+\infty,\boldsymbol{\infty})]}\big]^a \times \big[\mathcal{W}_{[(+\infty,\boldsymbol{\infty}),(+\infty,\boldsymbol{x})]}\mathcal{W}_{[(+\infty,\boldsymbol{x}),(x^0,\boldsymbol{x})]}E_i(x)\big]^a\Big\rangle_T$$

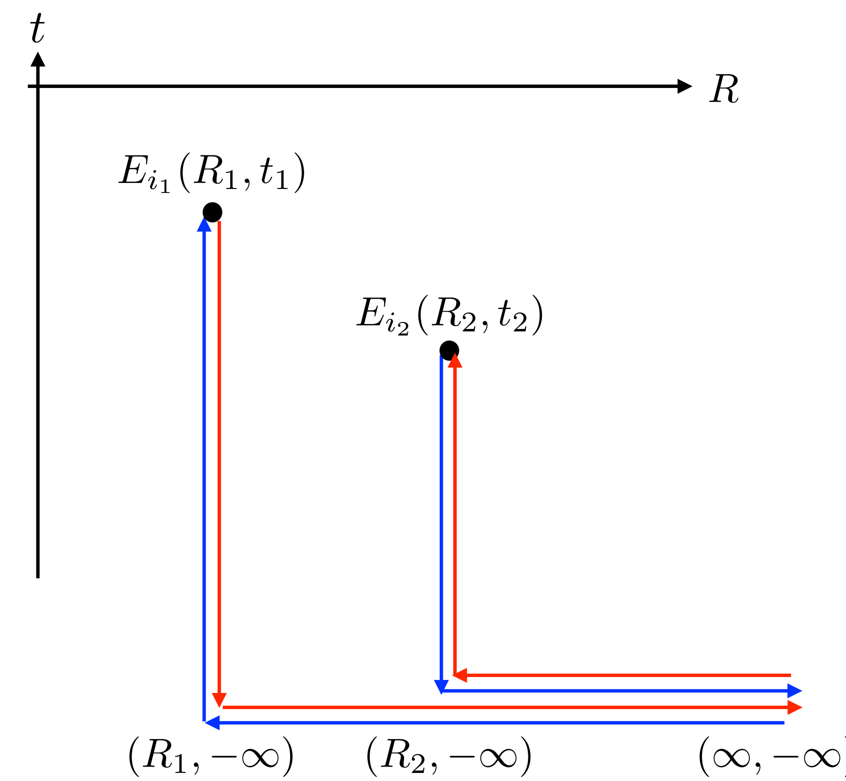
$$[g_E^{--}]^{>}_{ji}(y,x) \equiv \Big\langle \big[\mathcal{W}_{[(-\infty,\boldsymbol{\infty}),(-\infty,\boldsymbol{y})]}\mathcal{W}_{[(-\infty,\boldsymbol{y}),(y^0,\boldsymbol{y})]}E_j(y)\big]^a \times \big[E_i(x)\mathcal{W}_{[(x^0,\boldsymbol{x}),(-\infty,\boldsymbol{x})]}\mathcal{W}_{[(-\infty,\boldsymbol{x}),(-\infty,\boldsymbol{\infty})]}\big]^a\Big\rangle_T$$



PT transformation,
assume state invariant

⟷

KMS relation
for thermalization



Dissociation: final-state interaction

Recombination: initial-state interaction

For total reaction rates, integrating over final momentum gives setting $R_1 \to R_2$, the correlator becomes momentum independent

Backup: NLO Calculation in Real-Time Formalism

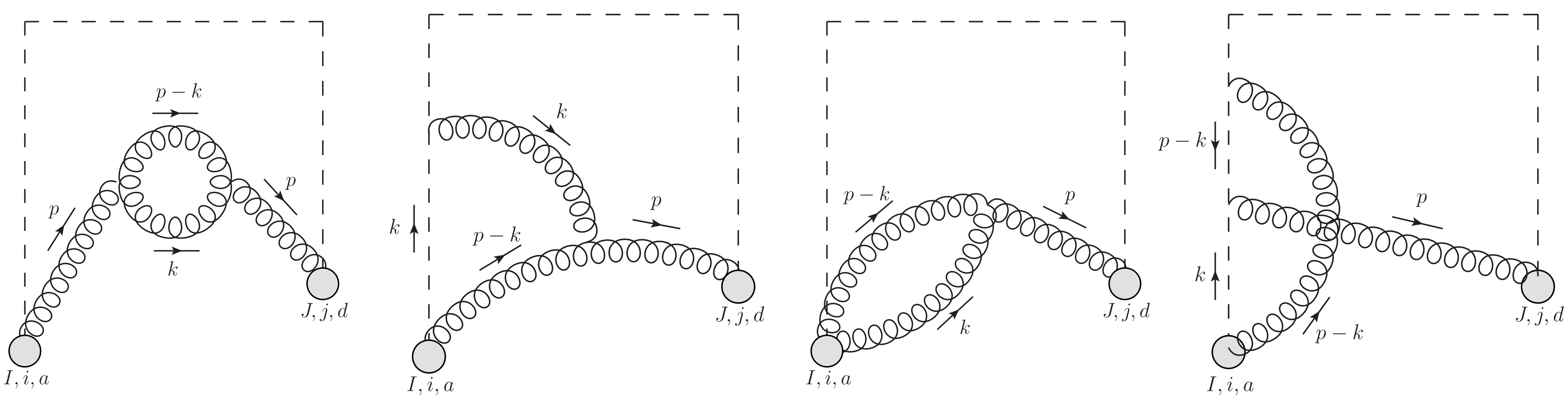
$$[g_E^{++}]^{>}_{ji}(y,x) \equiv \Big\langle \overbrace{\big[E_j(y)\mathcal{W}_{[(y^0,\boldsymbol{y}),(+\infty,\boldsymbol{y})]}\mathcal{W}_{[(+\infty,\boldsymbol{y}),(+\infty,\boldsymbol{\infty})]}\big]^a}^{\text{Type2}} \underbrace{\big[\mathcal{W}_{[(+\infty,\boldsymbol{\infty}),(+\infty,\boldsymbol{x})]}\mathcal{W}_{[(+\infty,\boldsymbol{x}),(x^0,\boldsymbol{x})]}E_i(x)\big]^a}_{\text{Type1}}\Big\rangle_T$$

Spectral function $$[\rho_E^{++}]_{ji}(y,x) \equiv [g_E^{++}]^{>}_{ji}(y,x) - [g_E^{++}]^{<}_{ji}(y,x)$$

Not able to put on standard Schwinger-Keldysh contour

$$[g_E^{++}]^{<}_{ji}(y,x) \equiv \Big\langle \big[\mathcal{W}_{[(+\infty,\boldsymbol{\infty}),(+\infty,\boldsymbol{x})]}\mathcal{W}_{[(+\infty,\boldsymbol{x}),(x^0,\boldsymbol{x})]}E_i(x)\big]^a \big[E_j(y)\mathcal{W}_{[(y^0,\boldsymbol{y}),(+\infty,\boldsymbol{y})]}\mathcal{W}_{[(+\infty,\boldsymbol{y}),(+\infty,\boldsymbol{\infty})]}\big]^a\Big\rangle_T$$

Replace $[g_E^{++}]^{<}_{ji}(y,x)$ **with** $[g_E^{--}]^{>}_{ji}(-y,-x)$



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