TMDs at sub-leading power "twist-3"

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Our results are PRELIMINARY and subject to change

New Mexico CONSORTIUM

U.S. DEPARTMENT OF ENERGY | Office of Science

TMD Collaboration

Motivation of my discussion/talk

- We explore subheading power TMDs in the context of factorization theorem
  - Relying on *TMD formalism* —extension of CSS, Abyat Rogers, Boer Pijlman Mulders-framework
  - Consider consistency of matching onto collinear factorization "revisit matching" see Bacchetta, Boer, Diehl, Mulders JHEP 2008 also in context of EOMs
  - Comment on recent work of MIT group, Gao, Ebert, Stewart 2021
  - Focus on Cahn effect & matching related to early picture of importance intrinsic $\mathbf{k}_T$
  - *INTRINSIC subheading twist TMDs—historical maybe not optimal*

Our results are PRELIMINARY and subject to change

Important papers

L, Gamberg, D Hwang, A Metz, M. Schlegel, Phys.Lett.B 639 (2006), hep-ph/0604022 [hep-ph]
A.P. Chen, J.P. Ma, Phys. Lett. B 768 (2017) 380, arXiv:1610 .08634.
I. Feige, D.W. Kolodrubetz, I. Moult, I.W. Stewart, J. High Energy Phys. 11 (2017) 142, arXiv:1703 .03411.
I. Balitsky, A. Tarasov, J. High Energy Phys. 07 (2017) 095, arXiv:1706 .01415.
I. Balitsky, A. Tarasov, J. High Energy Phys. 05 (2018) 150, arXiv:1712 .09389.
M.A. Ebert, I. Moult, I.W. Stewart, F.J. Tackmann, G. Vita, H.X. Zhu, J. High Energy Phys. 12 (2018) 084, arXiv:1807.10764.
M.A. Ebert, I. Moult, I.W. Stewart, F.J. Tackmann, G. Vita, H.X. Zhu, J. High Energy Phys. 04 (2019) 123, arXiv:1812 .08189
Moult, I.W. Stewart, G. Vita, arXiv:1905 .07411, 201
A. Bacchetta et al. / Physics Letters B 797 (2019) 134850
M. Ebert A. Gao I. Stewart arXiv:2112.07680

Why TMDs @ twist-3 → NLP

History

- Georgi Politzer, PRL 1978,
  "Clean Tests of QCD",
  QCD analysis of hard gluon radiation in SIDIS to predict absolute value of P_T
  & the angular distribution relative to lepton scattering plane
  "...angular correlations should be insensitive to nonperturbative effects."
- Cahn, PLB 1978, also earlier Ravndal, PLB 1972
  "Critique of the parton model calculation of azimuthal dependence in leptoproduction",
  importance intrinsic k_T ...
  "...The results can doubt on the utility of such azimuthal asymmetry as a clean test of quantum chromodynamics"

Clean tests of QCD

PHYSICAL REVIEW LETTERS

VOLUME 40 | 2 JANUARY 1978 | NUMBER 1

Clean Tests of Quantum Chromodynamics in μp Scattering

Howard Georgi
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138

and

H. David Politzer
California Institute of Technology, Pasadena, California 91125

(Received 25 October 1977)

Hard gluon bremsstrahlung in μp scattering produces final-state hadrons with a large component of momentum transverse to the virtual-photon direction. Quantum chromodynamics can be used to predict not only the absolute value of the transverse momentum, but also its angular distribution relative to the muon scattering plane. The angular correlations should be insensitive to nonperturbative effects.

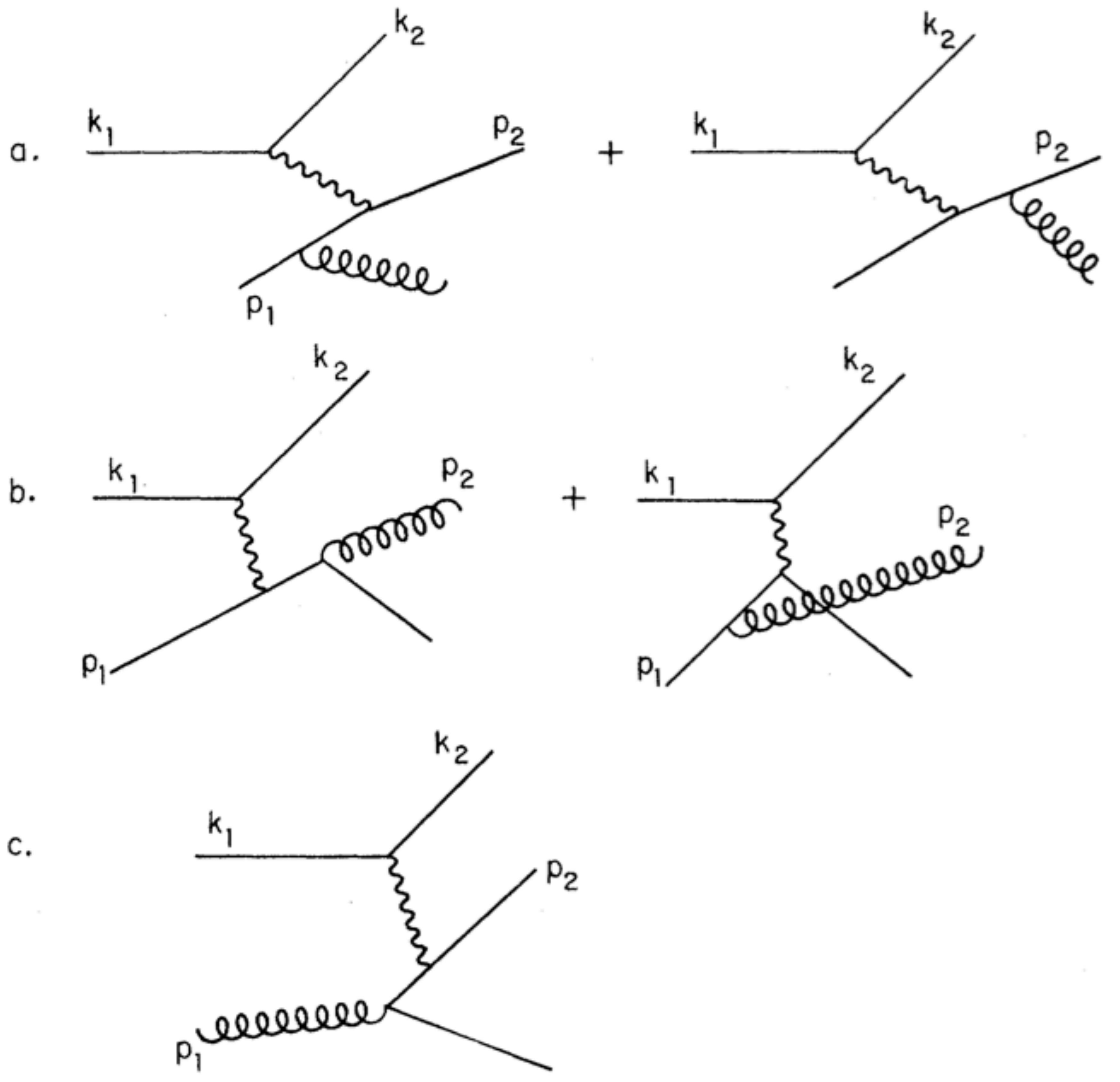


FIG. 1. Diagrams contributing to semi-inclusive μ-parton scattering to first order in α_s. k (p) denotes muon (parton) momentum. The wavy line is a virtual photon. The curly line is a gluon.

Cahn intrinsic k_T

Volume 78B, number 2,3 PHYSICS LETTERS 25 September 1978

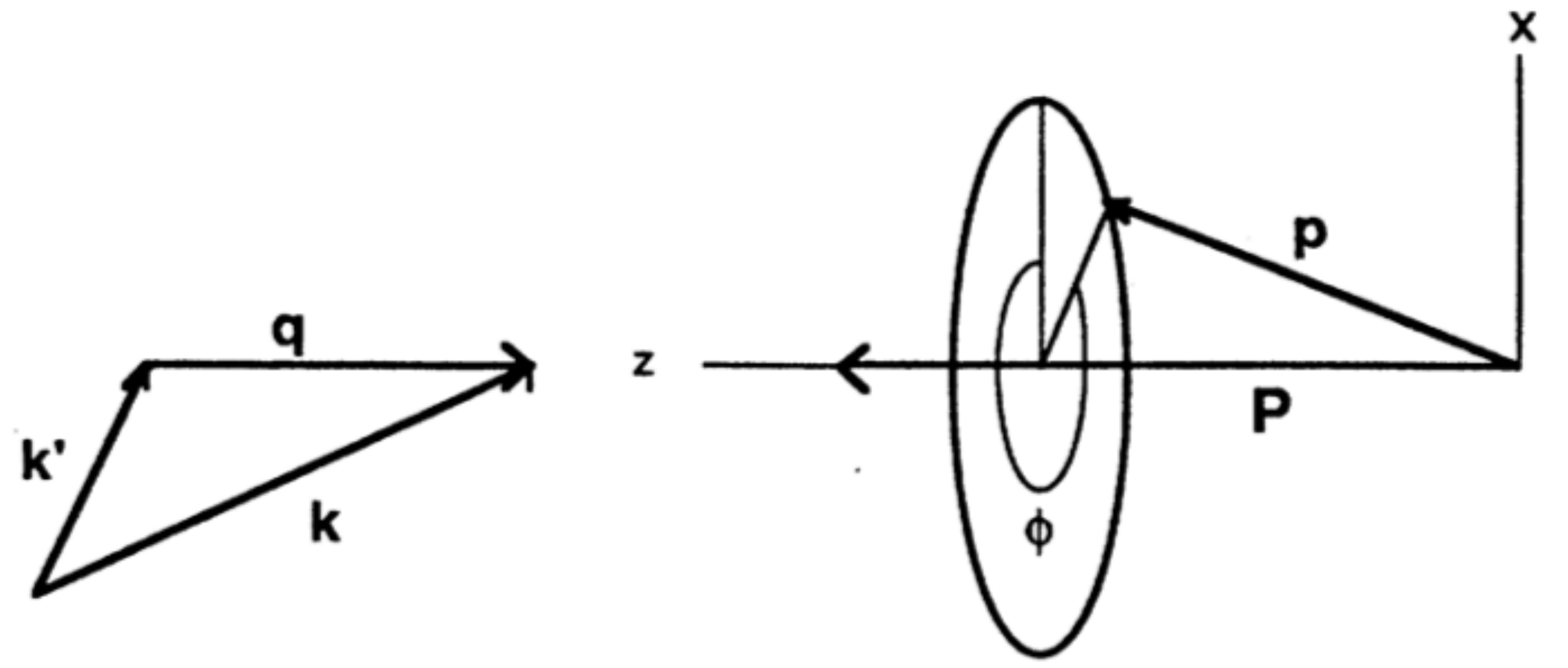
AZIMUTHAL DEPENDENCE IN LEPTOPRODUCTION: A SIMPLE PARTON MODEL CALCULATION☆

Robert N. CAHN
Department of Physics, University of Michigan, Ann Arbor, MI 48109, USA

Received 5 June 1978

Semi-inclusive leptoproduction, $\ell + p \rightarrow \ell' + h + X$, is considered in the naive parton model. The scattered parton shows an azimuthal asymmetry about the momentum transfer direction. Simple derivations for the effects in ep, νp and $\bar{\nu}$p scattering are given. Reduction of the asymmetry due to fragmentation of partons into hadrons is estimated. The results cast doubt on the utility of such azimuthal asymmetry as a clean test of quantum chromodynamics.

Cahn intrinsic k_T

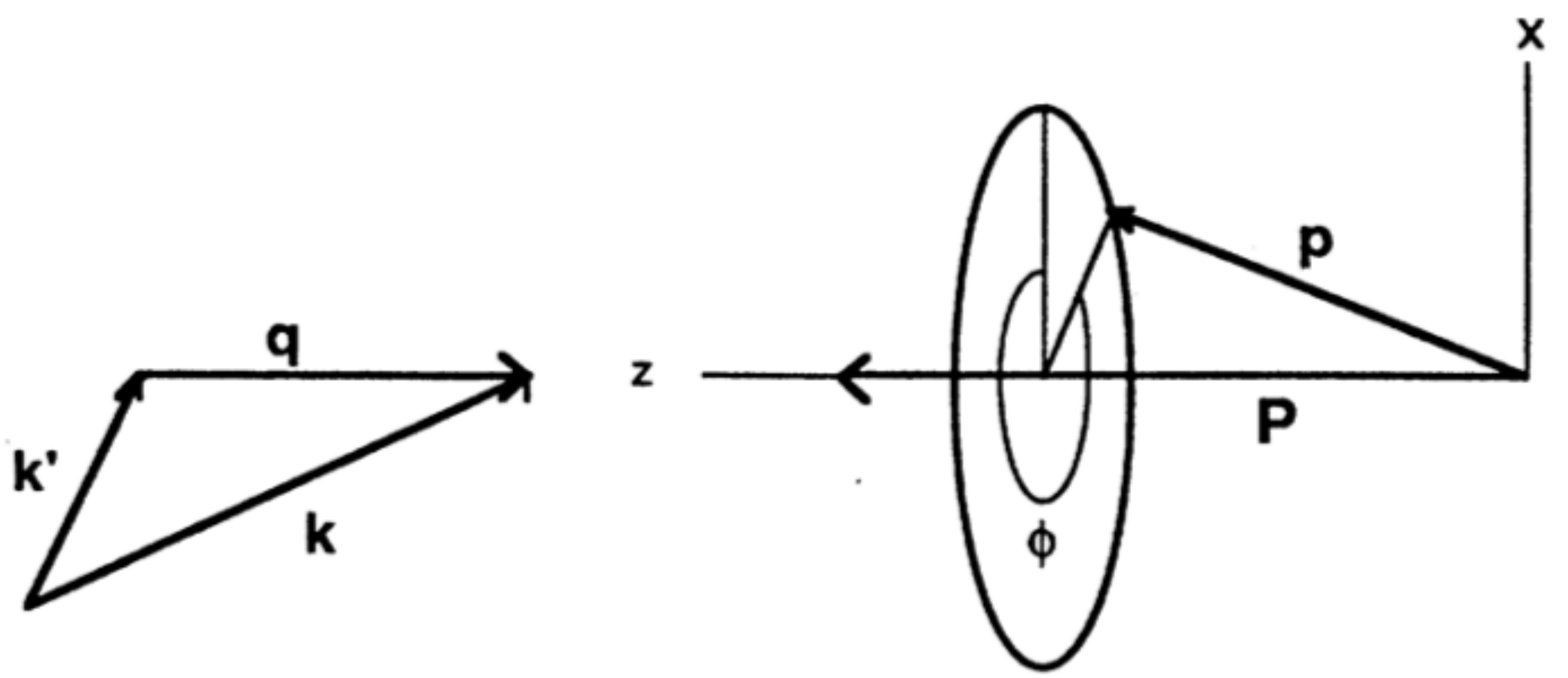


$$\sigma_{ep} \propto \hat{s}^2 + \hat{u}^2 \propto \left[1 - \frac{2p_\perp}{Q}\sqrt{1-y}\ \cos\phi\right]^2 + (1-y)^2\left[1 - \frac{2p_\perp}{Q\sqrt{1-y}}\cos\phi\right]^2$$

$$\langle \cos\phi \rangle_{ep} = -\left[\frac{2p_\perp}{Q}\right]\frac{(2-y)\sqrt{1-y}}{1+(1-y)^2}$$

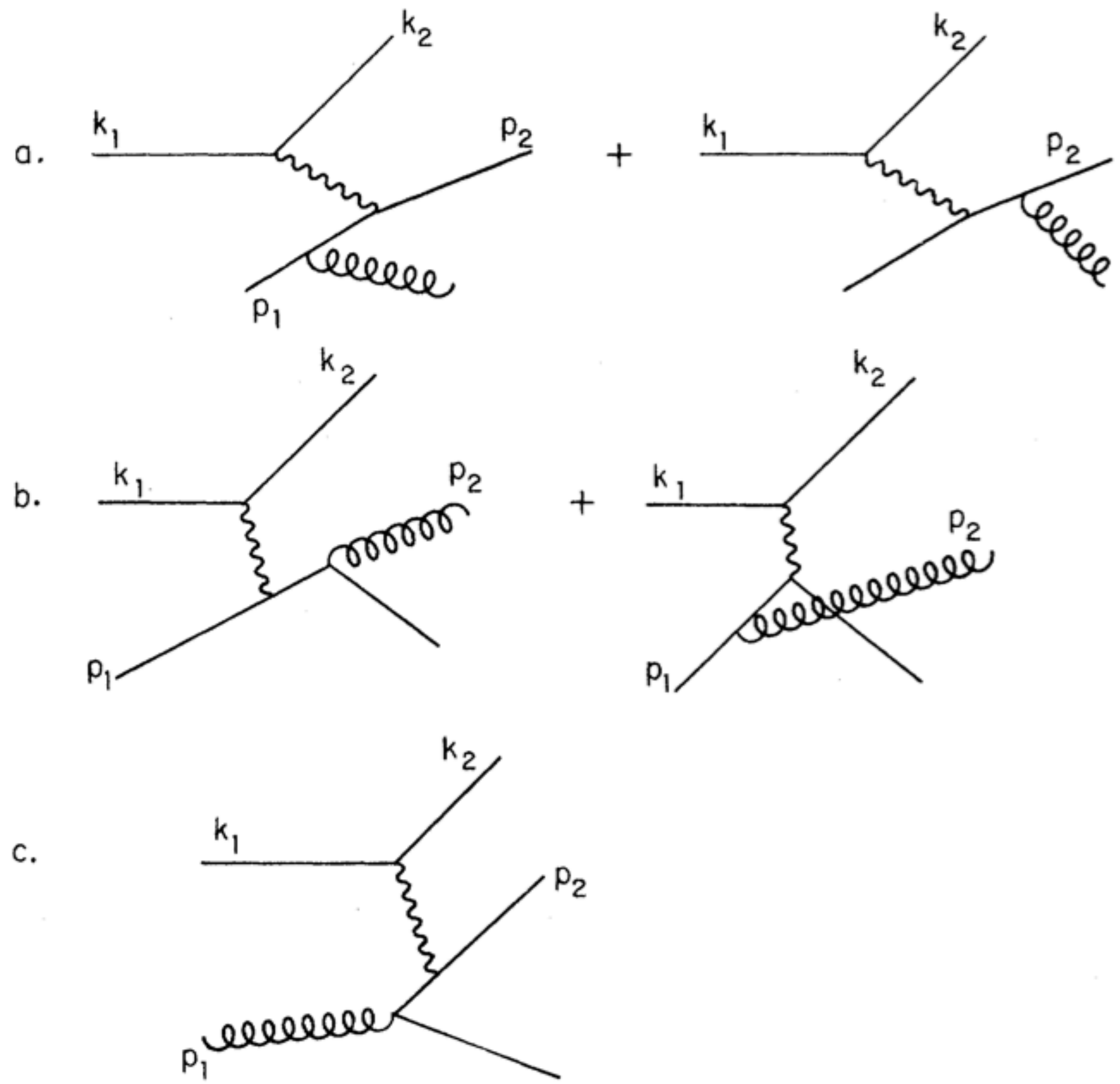
Tree level factorization sub-leading power

Cahn intrinsic $\boldsymbol{k}_T$



- "TMD" region

  $(p_T \sim k_T) \sim q_T \ll Q$



- "Collinear" region

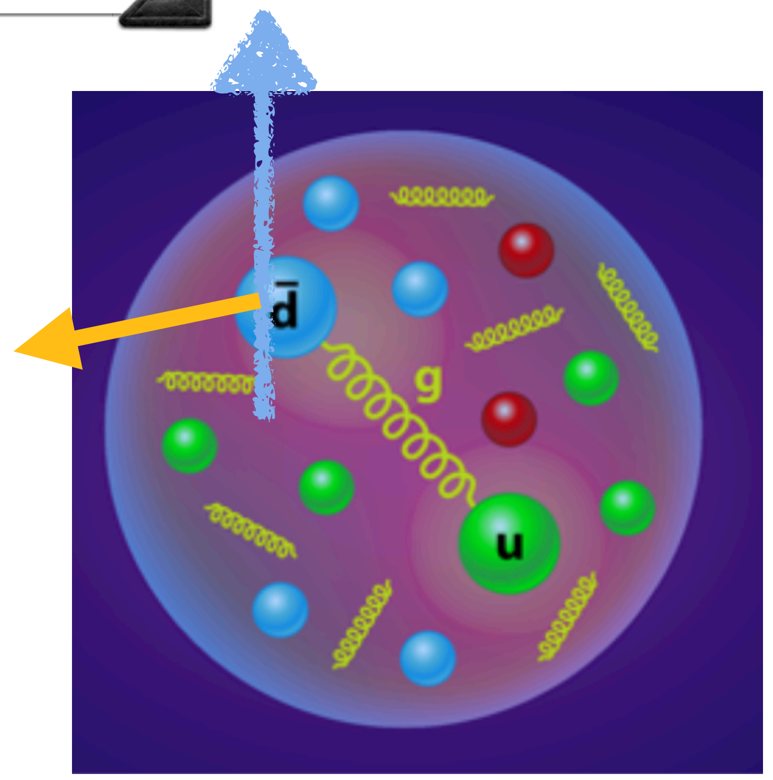
  $\Lambda_{qcd} \ll q_T \sim Q$

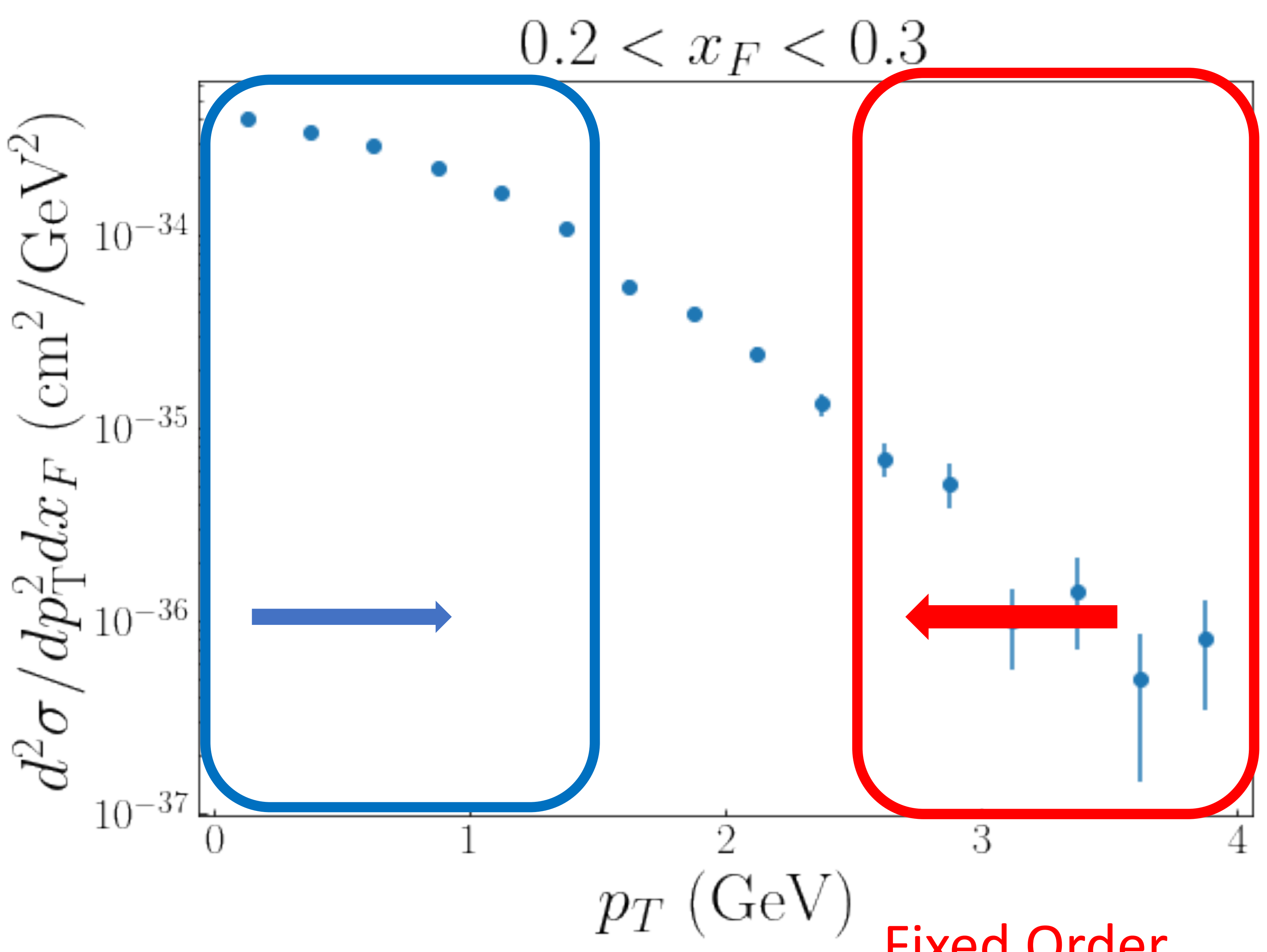
"More granular" $\Lambda_{QCD} \ll k_T \ll Q$ matching the TMD w/ the collinear factorization

To describe the asymptotic "region" $\Lambda_{QCD} \ll k_T \ll Q$ is the subject of "matching"

differential SIDS/ Drell-Yan cross section/ e^+e^- CSS NPB 1985, Catani et. al., $W + Y$ formalism-*unpolarized*

Bacchetta Boer Diehl Mulders (BBDM) matches &mismatches JHEP 2008 azimuthal & leading subheading power





TMD
Factorization

Fixed Order
Collinear
Factorization

Matching
TMD Factorization and collinear factorization

Overview comments Matching

- We modify the "*standard matching prescription*" traditionally used in CSS formalism relating low & high q_T behavior cross section @ moderate Q in particular where studies of TMDs are relevant

Matching studies in CSS related approaches

…

NPB Collins & Soper(1982), & Sterman 1985

NPB (1991) Arnold, Kauffman

PRD (1998) Nadolsky Stump Yuan

PRL (2001) Qiu, Zhang

PRD (2003) Berger, Qiu

NPB (2006) Bozzi, Catani, DeFlorian, Grazzini …

NPB (2006) Y. Koike, J. Nagashima, W. Vogelsang

JHEP (2008) Bacchetta et al.

arXiv (2014) Sun, Isacson, Yuan-CP,Yuan-F

JHEP (2015) Boglione, Hernandez, Melis Prokudin

PRD (2016) Collins, Gamberg, Prokudin, Rogers, Sato, Wang

PLB (2018) Gamberg , Metz, Pitonyak, Prokudin

PLB (2018) Echevarria, Kasemets, Lansberg, Pisano, Signori

EJPA (2018) Scimemi, Vladimirov

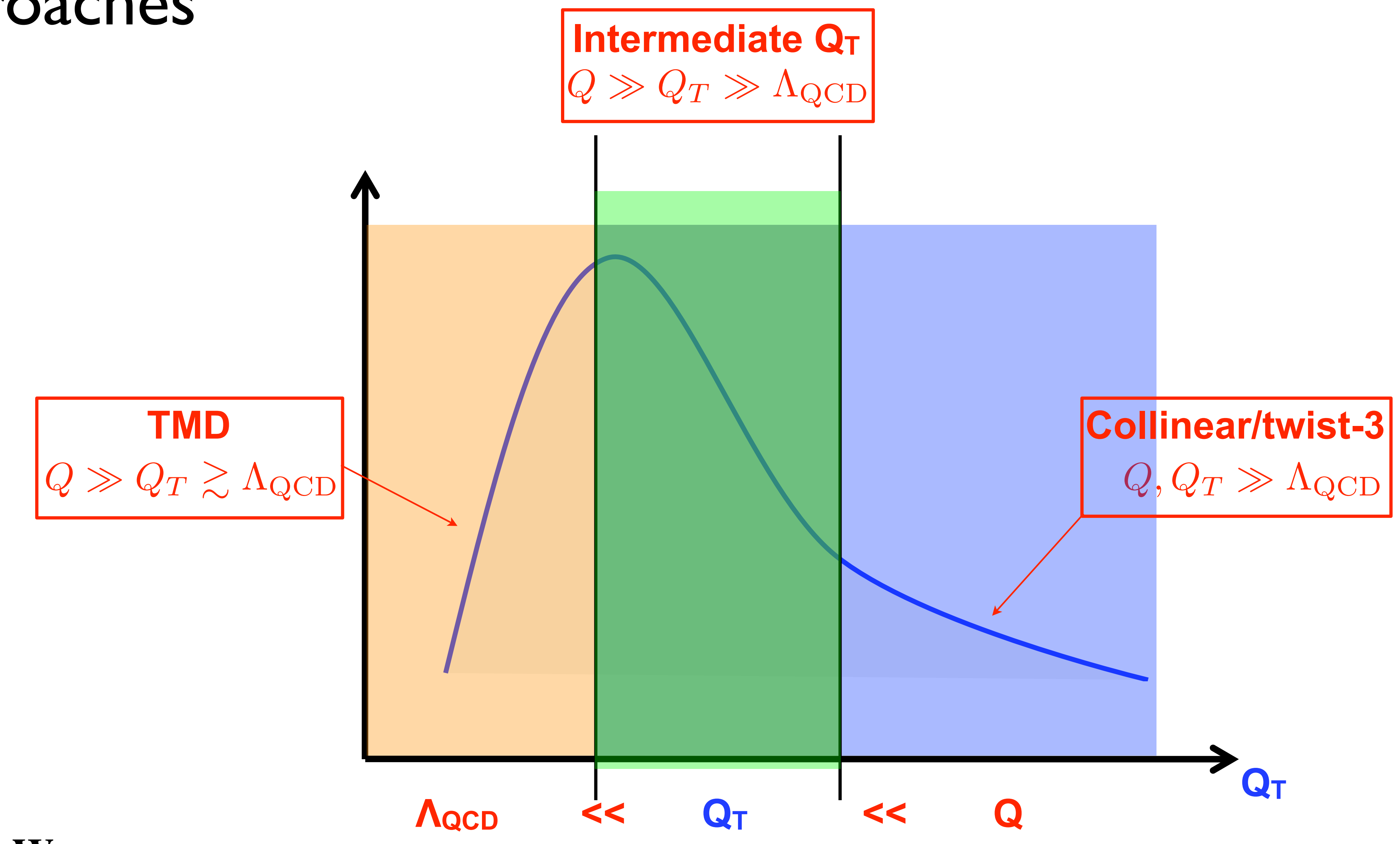
arXiv:1901.0451 Scimemi , Tarasov, Vladimirov

….

Series of papers on matching TMD and collinear ETQS transv. Spin

Ji, Qiu, Vogelsang, Yuan PRL PRD 2006, …

Kang, Xiao, Yuan PRL 2011



One finds the definition of the Y term via "approximators" CSS

$$Y(q_T, Q) \equiv T_{coll}\, d\sigma(q_T, Q) - T_{coll} T_{TMD}\, d\sigma(q_T, Q)$$

$$Y(q_T, Q) = FO(q_T, Q) - ASY(q_T, Q)$$

- It is the difference of the cross section calculated with collinear pdfs and ffs at fixed order FO and the asymptotic contribution of the cross section

- ***nb at small q_T the FO and ASY are dominated by the same diverging terms***

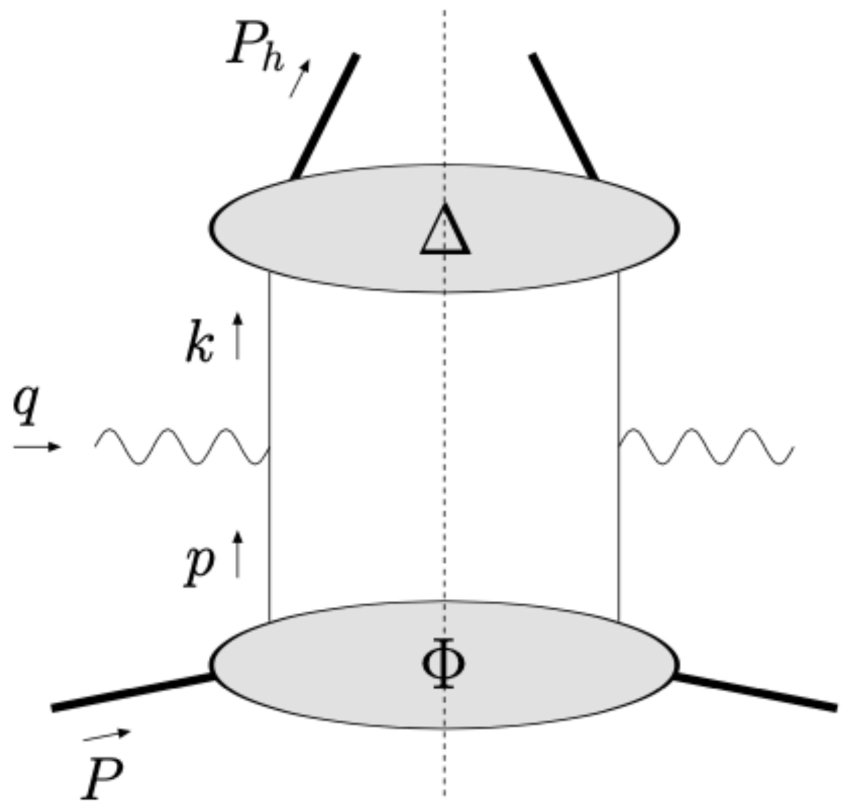
$$\frac{1}{q_T^2} \quad \text{and} \quad \frac{1}{q_T^2} \log \frac{Q^2}{q_T^2}$$

- ***Thus its expected that the Y term is small or zero leaving***

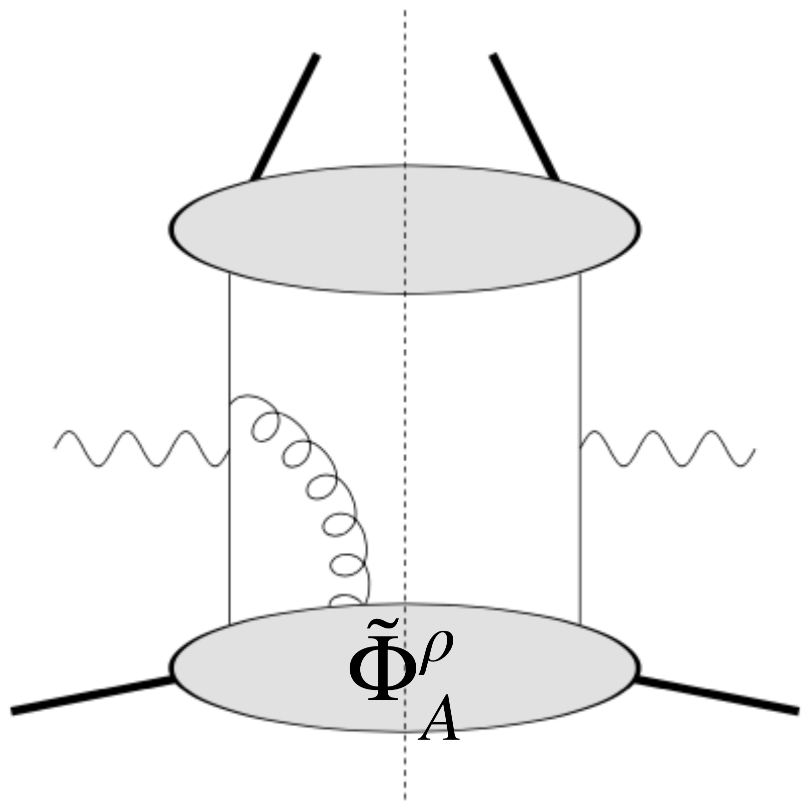
$$d\sigma(q_T \ll Q, Q) \approx W(q_T, Q)$$

To setup our approach to factorization at sub-leading power … revisit Tree level

- "TMD" region
  $(p_T \sim k_T) \sim q_T \ll Q$



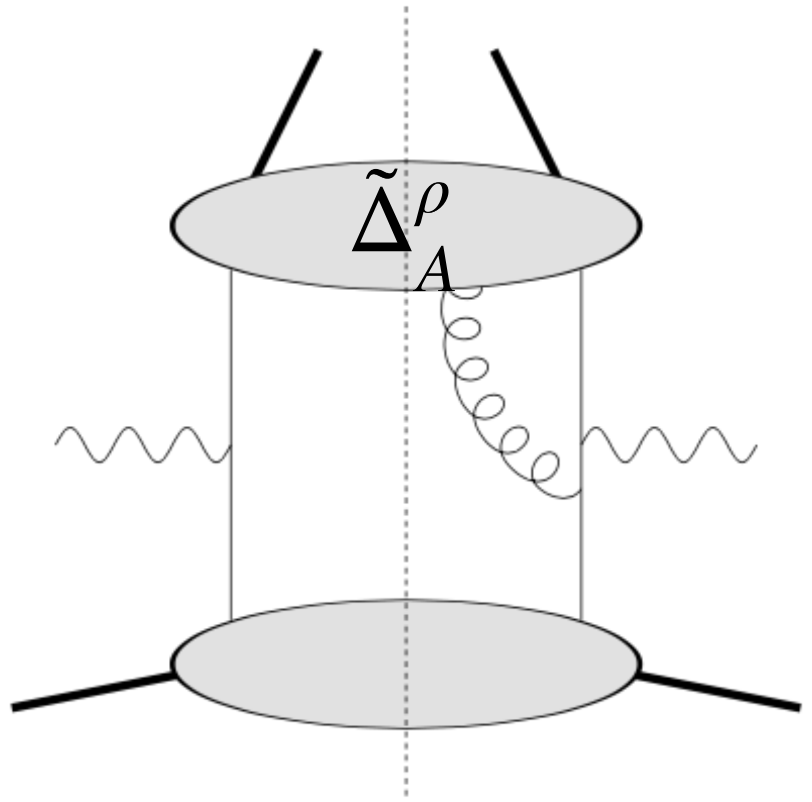
(a)



(b)

SIDIS tree-level diagrams relevant for sub-leading-power observables. Upper left contain *"intrinsic and kinematical"* contributions, the other two diagrams "*dynamical*" contributions with

$$A_T = n_T \cdot A$$



(c)

$$
\begin{aligned}
2M\mathcal{W}^{\mu\nu} =& e^2 \int d^2\boldsymbol{p}_T d^2\boldsymbol{k}_T \delta^2(\boldsymbol{p}_T + \boldsymbol{q}_T - \boldsymbol{k}_T)\Big\{ \mathrm{Tr}\left[\Phi(p)\gamma_\mu \Delta(k)\gamma_\nu\right] \\
&- \frac{1}{Q2\sqrt{2}}\mathrm{Tr}\left[\gamma_\alpha \not{\bar{n}} \gamma_\nu \Phi_A^\alpha(p)\gamma_\mu \Delta(k)\right] - \frac{1}{Q2\sqrt{2}}\mathrm{Tr}\left[\gamma_\mu \not{\bar{n}} \gamma_\alpha \Delta(k)\gamma_\nu \Phi_A^{\alpha\dagger}(p)\right] \\
&- \frac{1}{Q2\sqrt{2}}\mathrm{Tr}\left[\gamma_\nu \not{n} \gamma_\alpha \Phi(p)\gamma_\mu \Delta_A^{\alpha\dagger}(k)\right] - \frac{1}{Q2\sqrt{2}}\mathrm{Tr}\left[\gamma_\alpha \not{n} \gamma_\mu \Delta_A^\alpha(k)\gamma_\nu \Phi(p)\right] \Big\}
\end{aligned}
$$

Tree level factorization sub-leading power

$$\Phi(x, k_T)$$

SIDIS tree-level diagrams relevant for sub-leading-power observables.
"intrinsic"

✦Mulders Tangerman NPB1995
✦Goeke Metz Schlegel PLB 2005
✦Bacchetta et al 2007 JHEP

Subleading Quark TMDPDFs

Nucleon Polarization	Quark Chirality: Chiral Even	Quark Chirality: Chiral Odd
U	$f^{\perp}, g^{\perp}$	$e\,,\ h$
L	$f_L^{\perp}, g_L^{\perp}$	e_L, h_L
T	$f_T, f_T^{\perp}, g_T, g_T^{\perp}$	$e_T, e_T^{\perp}, h_T, h_T^{\perp}$

$$\begin{aligned}\Phi^{(3)}(x,\boldsymbol{k}_T,\boldsymbol{S}) = \frac{M}{P^+}\Bigg[&\left(e - \frac{\epsilon_T^{\rho\sigma} k_{T\rho} S_{T\sigma}}{M} e_T^{\perp}\right)\frac{1}{2} - i\left(\lambda_g e_L - \frac{\boldsymbol{k}_T\cdot\boldsymbol{S}_T}{M} e_T\right)\frac{\gamma^5}{2} \\ &+ \left(\frac{k_T^k}{M} f^{\perp} - \epsilon_T^{kl} S_{Tl} f_T' - \frac{\epsilon_T^{kl} k_{Tl}}{M}\left(\lambda_g f_L^{\perp} - \frac{\boldsymbol{k}_T\cdot\boldsymbol{S}_T}{M} f_T^{\perp}\right)\right)\frac{\gamma_k}{2} \\ &+ \left(g_T' S_T^k - \frac{\epsilon_T^{kl} k_{Tl}}{M} g^{\perp} + \frac{k_T^k}{M}\left(\lambda_g g_L^{\perp} - \frac{\boldsymbol{k}_T\cdot\boldsymbol{S}_T}{M} g_T^{\perp}\right)\right)\frac{\gamma^5\gamma_k}{2} \\ &+ \left(\frac{S_T^k k_T^l}{M} h_T^{\perp}\right)\frac{i\gamma^5\sigma_{lk}}{4} + \left(h + \lambda_g h - \frac{\boldsymbol{k}_T\cdot\boldsymbol{S}_T}{M} h^{\perp}\right)\frac{i\gamma^5\sigma_{+-}}{4}\Bigg]\end{aligned}$$

Tree level factorization sub-leading power

$$\tilde{\Phi}_A^{\rho}(x, \boldsymbol{k}_T)$$

SIDIS tree-level diagrams relevant for sub-leading-power observables.
Upper left contain *"intrinsic and kinematical"* contributions, the other two diagrams "*dynamical*" contributions with

$$A_T = n_T \cdot A$$

- ✦Mulders Tangerman NPB1995
- ✦ Boer Pijlman Mulders NPB 2003
- ✦Bacchetta et al 2007 JHEP

Subleading Quark-Gluon-Quark TMDPDFs

		Quark Chirality	
		Chiral Even	Chiral Odd
Nucleon Polarization	U	$\tilde{f}^{\perp}, \tilde{g}^{\perp}$	$\tilde{e}\,,\ \tilde{h}$
	L	$\tilde{f}_L^{\perp}, \tilde{g}_L^{\perp}$	$\tilde{e}_L, \tilde{h}_L$
	T	$\tilde{f}_T, \tilde{f}_T^{\perp}, \tilde{g}_T, \tilde{g}_T^{\perp}$	$\tilde{e}_T, \tilde{e}_T^{\perp}, \tilde{h}_T, \tilde{h}_T^{\perp}$

$$\tilde{\Phi}_A^{\alpha}(x, p_T) =$$

$$\frac{xM}{2}\left\{\left[(\tilde{f}^{\perp} - i\tilde{g}^{\perp})\frac{p_{T\rho}}{M} - (\tilde{f}_T' + i\tilde{g}_T')\,\epsilon_{T\rho\sigma}S_T^{\sigma} - (\tilde{f}_s^{\perp} + i\,\tilde{g}_s^{\perp})\frac{\epsilon_{T\rho\sigma}p_T^{\sigma}}{M}\right](g_T^{\alpha\rho} - i\epsilon_T^{\alpha\rho}\gamma_5)\right.$$

$$\left. - (\tilde{h}_s + i\,\tilde{e}_s)\gamma_T^{\alpha}\gamma_5 + \left[(\tilde{h} + i\,\tilde{e}) + (\tilde{h}_T^{\perp} - i\,\tilde{e}_T^{\perp})\frac{\epsilon_T^{\rho\sigma}p_{T\rho}S_{T\sigma}}{M}\right]i\gamma_T^{\alpha} + \ldots (g_T^{\alpha\rho} + i\epsilon_T^{\alpha\rho}\gamma_5)\right\}\frac{\not{n}_+}{2}$$

Tree level factorization sub-leading power

$$\Phi(x, p_T) = \frac{1}{4} f_1 \not{\bar{n}} + \frac{1}{2P^+} f^\perp \not{p}_T + \cdots ,$$

$$\Delta(z, k_T) = \frac{1}{4} D_1 \not{n} + \frac{1}{2P_h^-} D^\perp \not{k}_T + \cdots ,$$

$$\tilde{\Phi}_A^\alpha(x, p_T) = \frac{x p_{T\rho}}{4} \left(\tilde{f}^\perp - i\tilde{g}^\perp \right) \left(g_T^{\alpha\rho} - i\epsilon_T^{\alpha\rho} \right) \not{\bar{n}} + \cdots ,$$

$$\tilde{\Delta}_A^\alpha(z, k_T) = \frac{k_{T\rho}}{4z} \left(\tilde{D}^\perp - i\tilde{G}^\perp \right) \left(g_T^{\alpha\rho} + i\epsilon_T^{\alpha\rho} \right) \not{n} + \cdots .$$

Tree level factorization sub-leading power SIDIS

$$\Phi(x, p_T) = \frac{1}{4} f_1 \not{\bar{n}} + \frac{1}{2P^+} f^\perp \not{p}_T + \cdots \,,$$

$$\Delta(z, k_T) = \frac{1}{4} D_1 \not{n} + \frac{1}{2P_h^-} D^\perp \not{k}_T + \cdots \,,$$

$$\tilde{\Phi}_A^\alpha(x, p_T) = \frac{x p_{T\rho}}{4} \left(\tilde{f}^\perp - i\tilde{g}^\perp\right) \left(g_T^{\alpha\rho} - i\epsilon_T^{\alpha\rho}\right) \not{\bar{n}} + \cdots \,,$$

$$\tilde{\Delta}_A^\alpha(z, k_T) = \frac{k_{T\rho}}{4z} \left(\tilde{D}^\perp - i\tilde{G}^\perp\right) \left(g_T^{\alpha\rho} + i\epsilon_T^{\alpha\rho}\right) \not{n} + \cdots \,.$$

Using the EoM relations $x f^\perp = x\tilde{f}^\perp + f_1$ and $D^\perp/z = \tilde{D}^\perp/z + D_1$, one has

$$\mathrm{Tr}\left[\Phi(p)\gamma^\mu \Delta(k)\gamma^\nu\right] = -g_\perp^{\mu\nu} f_1 D_1 + \frac{\sqrt{2}}{2Q} \frac{f_1 \tilde{D}^\perp}{z} \bar{n}^{\{\mu} k_T^{\nu\}}$$

$$+ \frac{\sqrt{2}}{2Q} (\bar{n} + n)^{\{\mu} p_T^{\nu\}} x f^\perp D_1 - \frac{\sqrt{2}}{2Q} \bar{n}^{\{\mu} p_T^{\nu\}} x \tilde{f}^\perp D_1$$

Tree level factorization sub-leading power

$$\Phi(x,p_T) = \frac{1}{4} f_1 \bar{\not{n}} + \frac{1}{2P^+} f^\perp \not{p}_T + \cdots\,,$$

$$\Delta(z,k_T) = \frac{1}{4} D_1 \not{n} + \frac{1}{2P_h^-} D^\perp \not{k}_T + \cdots\,,$$

$$\tilde{\Phi}_A^\alpha(x,p_T) = \frac{x p_{T\rho}}{4} \left(\tilde{f}^\perp - i\tilde{g}^\perp\right) \left(g_T^{\alpha\rho} - i\epsilon_T^{\alpha\rho}\right) \bar{\not{n}} + \cdots\,,$$

$$\tilde{\Delta}_A^\alpha(z,k_T) = \frac{k_{T\rho}}{4z} \left(\tilde{D}^\perp - i\tilde{G}^\perp\right) \left(g_T^{\alpha\rho} + i\epsilon_T^{\alpha\rho}\right) \not{n} + \cdots\,.$$

From three parton correlators get dynamic contributions

$$\begin{aligned}
&- \frac{1}{Q2\sqrt{2}} \mathrm{Tr}\left[\gamma_\alpha \bar{\not{n}} \gamma_\nu \Phi_A^\alpha(p) \gamma_\mu \Delta(k)\right] - \frac{1}{Q2\sqrt{2}} \mathrm{Tr}\left[\gamma_\mu \bar{\not{n}} \gamma_\alpha \Delta(k) \gamma_\nu \Phi_A^{\alpha\dagger}(p)\right] \\
&- \frac{1}{Q2\sqrt{2}} \mathrm{Tr}\left[\gamma_\nu \not{n} \gamma_\alpha \Phi(p) \gamma_\mu \Delta_A^{\alpha\dagger}(k)\right] - \frac{1}{Q2\sqrt{2}} \mathrm{Tr}\left[\gamma_\alpha \not{n} \gamma_\mu \Delta_A^\alpha(k) \gamma_\nu \Phi(p)\right] \\
&= \frac{\sqrt{2}}{2Q} \bar{n}^{\{\mu} p_T^{\nu\}} x \tilde{f}^\perp D_1 + \frac{\sqrt{2}}{2Q} n^{\{\mu} k_T^{\nu\}} \frac{f_1 \tilde{D}^\perp}{z}
\end{aligned}$$

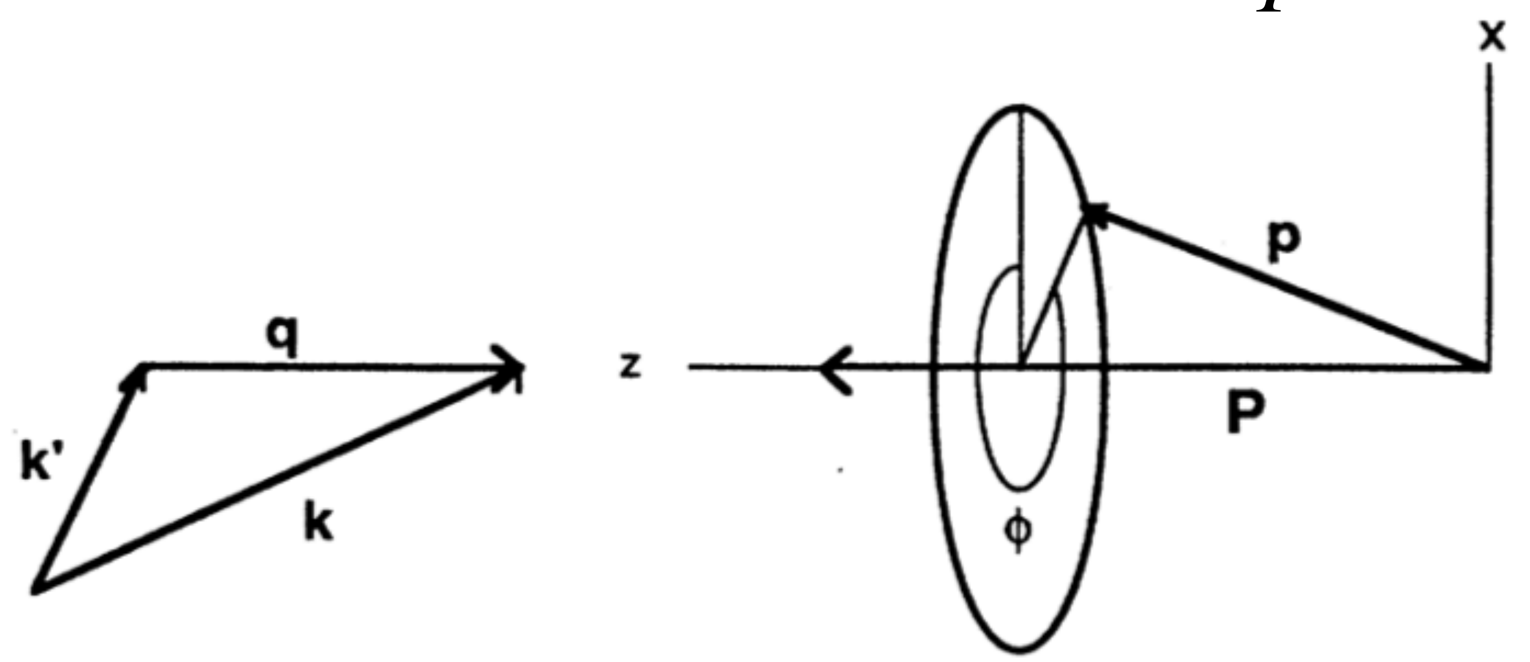
Tree level factorization sub-leading power

Combining these contributions and multiplying by leptonic tensor
get Cahn and more ….

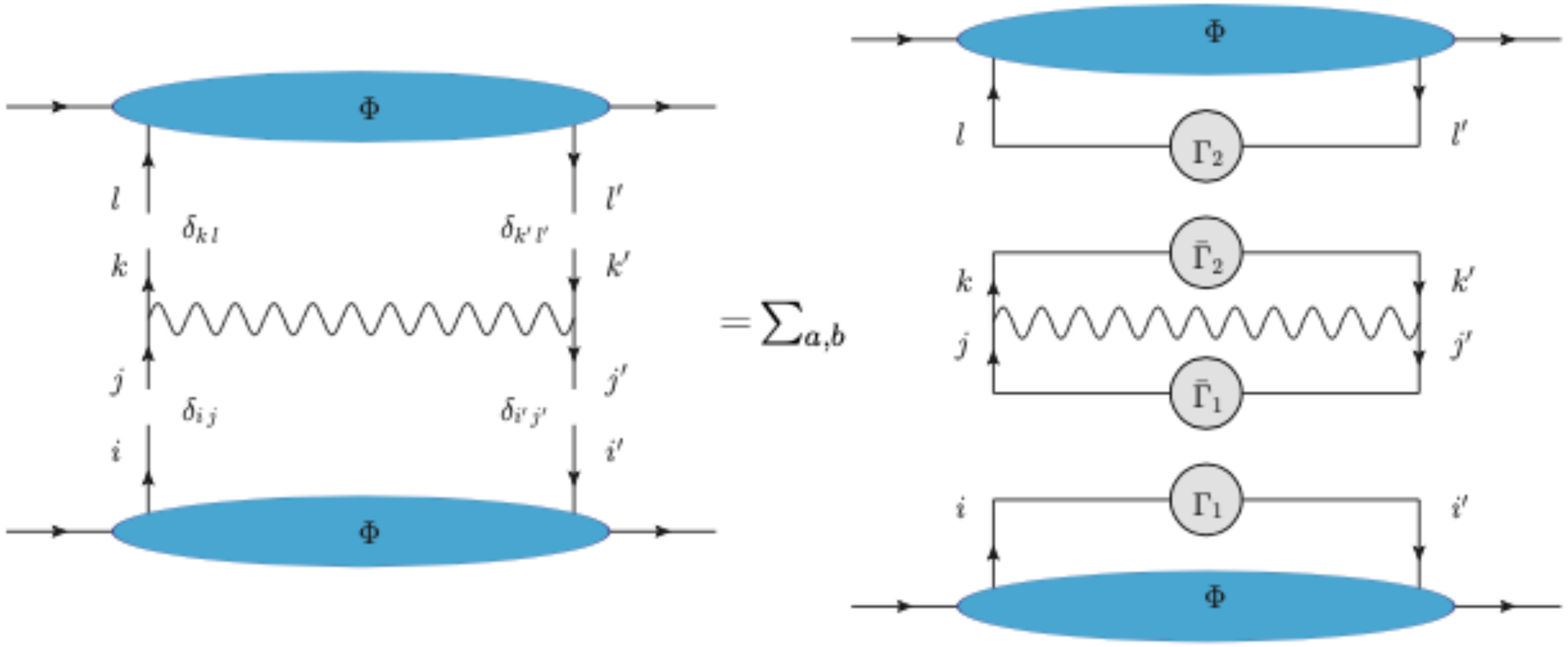
$$\frac{1}{Q}\hat{t}^{\{\mu}k_T^{\nu\}}\frac{f_1\tilde{D}^{\perp}}{z}L_{\mu\nu} = -\frac{4Q^2}{y^2}(2-y)\sqrt{1-y}\left[\frac{1}{Q}\,\hat{\boldsymbol{h}}\cdot\boldsymbol{k}_T\frac{f_1\tilde{D}^{\perp}}{z}\right]\cos\phi_h$$

$$\frac{2}{Q}\hat{t}^{\{\mu}p_T^{\nu\}}xf^{\perp}D_1L_{\mu\nu} = -\frac{4Q^2}{y^2}(2-y)\sqrt{1-y}\left[\frac{2}{Q}\,\hat{\boldsymbol{h}}\cdot\boldsymbol{p}_T xf^{\perp}D_1\right]\cos\phi_h$$

Cahn intrinsic $\boldsymbol{k}_T$



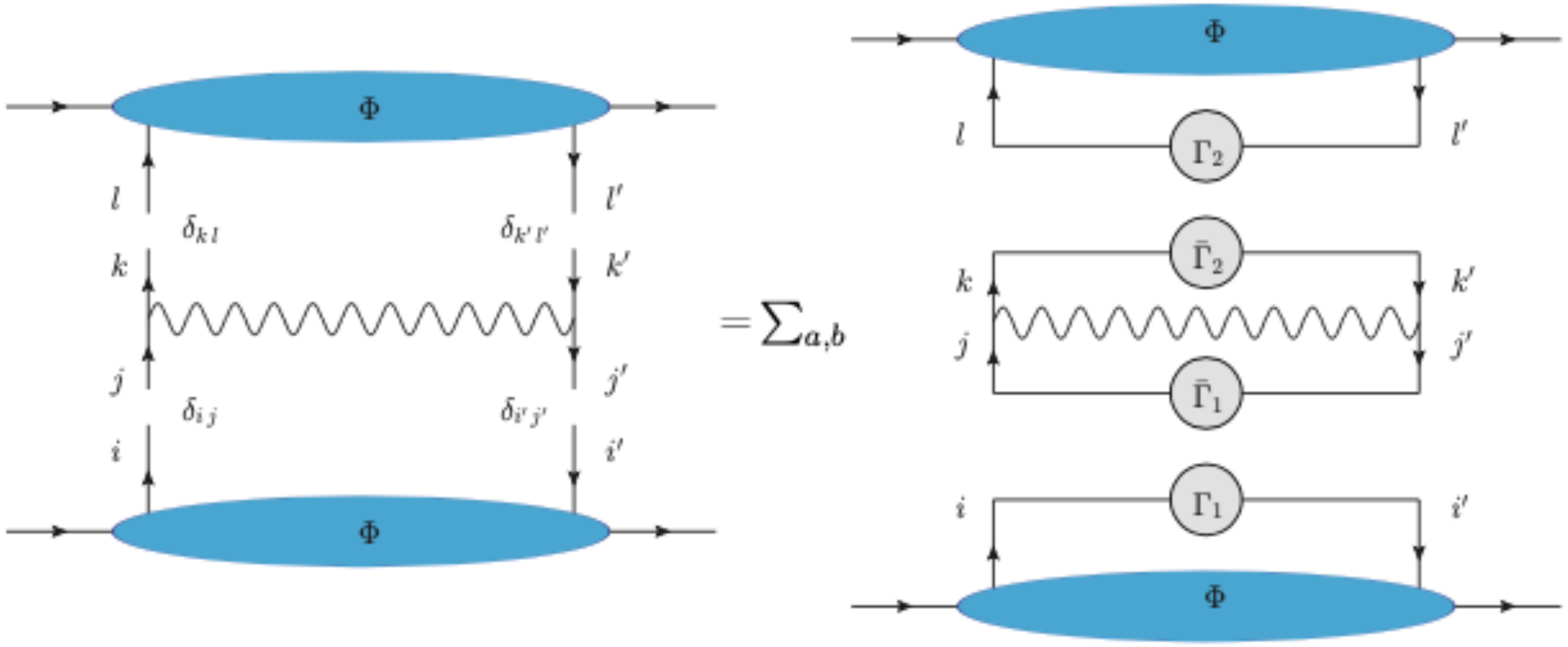
Factorization for subheading power via Fierz decomp motivate TMD factorization framework



Twist 2	Twist 3	Twist 4
$\frac{1}{2}\not{n}$, $\frac{1}{4}\bar{\not{n}}$	$\frac{1}{2},\frac{1}{2}$	$\frac{1}{2}\bar{\not{n}}$, $\frac{1}{4}\not{n}$
$\frac{1}{2}\not{n}\gamma^5$, $\frac{1}{4}\gamma^5\bar{\not{n}}$	$\frac{1}{2}\gamma^5$, $\frac{1}{2}\gamma^5$	$\bar{\not{n}}\gamma^5$, $\frac{1}{4}\gamma^5\not{n}$
$\frac{i}{2}\sigma^{k+}\gamma^5$, $\frac{i}{4}\gamma^5\sigma_{-k}$	$\frac{1}{2}\gamma^k$, $\frac{1}{2}\gamma_k$	$\frac{i}{2}\sigma^{k-}\gamma^5$, $\frac{i}{4}\gamma^5\sigma_{+k}$
	$\frac{1}{2}\gamma^k\gamma^5$, $\frac{1}{2}\gamma^5\gamma_k$	
	$\frac{i}{2}\sigma^{kl}\gamma^5$, $\frac{i}{4}\gamma^5\sigma_{lk}$	
	$\frac{i}{4}\sigma^{+-}\gamma^5$, $\frac{i}{4}\gamma^5\sigma_{+-}$	

$$\Phi^{\Gamma^a}\left(x_1, \boldsymbol{k}_T, \boldsymbol{S}\right) \equiv \mathrm{Tr}\left[\Phi\left(x_1, \boldsymbol{k}_T, \boldsymbol{S}\right)\Gamma^a\right]$$

Organize via Fierz decomp motivate TMD factorization framework



Twist 2	Twist 3	Twist 4
$\frac{1}{2}\not{n}\,,\ \frac{1}{4}\not{\bar{n}}$	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}\not{\bar{n}}\,,\ \frac{1}{4}\not{n}$
$\frac{1}{2}\not{n}\gamma^5,\ \frac{1}{4}\gamma^5\not{\bar{n}}$	$\frac{1}{2}\gamma^5,\ \frac{1}{2}\gamma^5$	$\frac{1}{2}\not{\bar{n}}\gamma^5,\ \frac{1}{4}\gamma^5\not{n}$
$\frac{i}{2}\sigma^{k+}\gamma^5,\ \frac{i}{4}\gamma^5\sigma_{-k}$	$\frac{1}{2}\gamma^k,\ \frac{1}{2}\gamma_k$	$\frac{i}{2}\sigma^{k-}\gamma^5\,,\ \frac{i}{4}\gamma^5\sigma_{+k}$
	$\frac{1}{2}\gamma^k\gamma^5,\ \frac{1}{2}\gamma^5\gamma_k$	
	$\frac{i}{2}\sigma^{kl}\gamma^5,\ \frac{i}{4}\gamma^5\sigma_{lk}$	
	$\frac{i}{4}\sigma^{+-}\gamma^5,\ \frac{i}{4}\gamma^5\sigma_{+-}$	

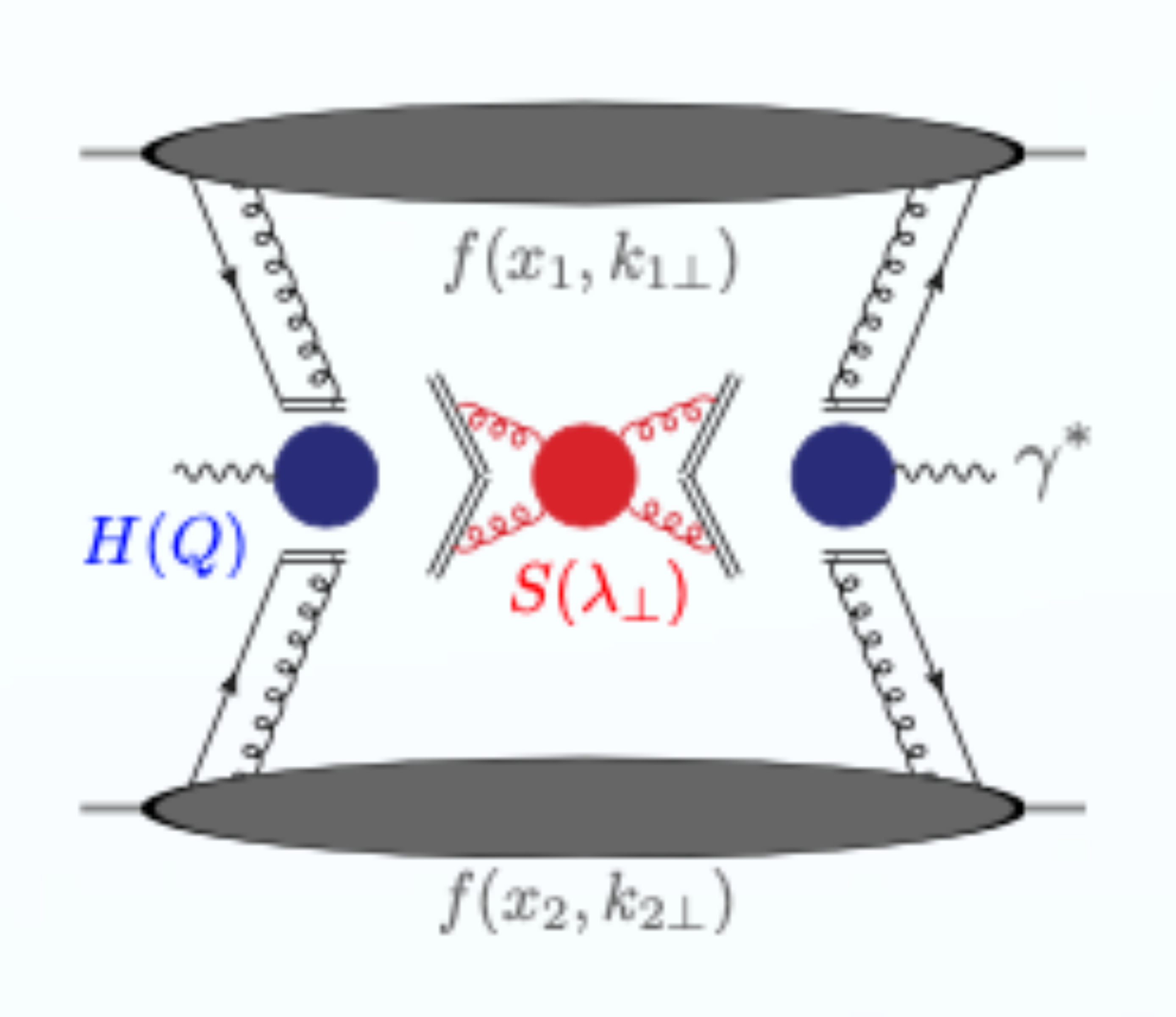
$$W^{\mu\nu}_{\{2,3\,\text{intrinsic}\}} = \frac{1}{N_c}\sum_{a1,a2}\sum_q e_q^2 \int d^2k_{1T}\, d^2k_{2T}\, \delta^{(2)}\left(\boldsymbol{q}_T - \boldsymbol{k}_{1T} - \boldsymbol{k}_{2T}\right)$$

$$\times \operatorname{Tr}\left[\gamma^\mu \bar{\Gamma}_1^{a1}\, \gamma^\nu\, \bar{\Gamma}_1^{a2}\right]\, \Phi^{[\Gamma^{a1}]}\left(x_1, \boldsymbol{k}_{1T}, \boldsymbol{S}_1\right)\, \bar{\Phi}^{[\Gamma^{a2}]}\left(x_2, \boldsymbol{k}_{2T}, \boldsymbol{S}_2\right).$$

$$\Phi^{\Gamma^a}\left(x_1, \boldsymbol{k}_T, \boldsymbol{S}\right) \equiv \operatorname{Tr}\left[\Phi\left(x_1, \boldsymbol{k}_T, \boldsymbol{S}\right)\Gamma^a\right]$$

Representation of the Fierz decompostion of the hadronic tensor.
Left: broken lines used to separate the hard interaction from the definition of the qq correlation function.
Right: The Fierz decomposition where Γ_a represent the operators which give rise to the parton densities while $\bar{\Gamma}_a$ represent the operators which enter into the hard function.

Extend TMD factorization, renormalization & evolution to sub leading power



$q_T \sim k_T \ll Q$

TMD Factorization

- ✦*Collins Soper Sterman NPB 1985*
- ✦*Ji Ma Yuan PRD PLB …2004, 2005*
- ✦*Aybat Rogers PRD 2011*
- ✦*Collins 2011 Cambridge Press*
- ✦*Echevarria, Idilbi, Scimemi JHEP 2012, …*
- ✦*SCET Becher & Neubert, 2011 EJPC*

$$\frac{d\sigma^W}{dQ^2\, dx_F\, dp_T^2} = \int \frac{d^2\boldsymbol{b}_T}{(2\pi)^2}\, e^{i\boldsymbol{p}_T\cdot\boldsymbol{b}_T}\tilde{W}(x_F, b_T, Q)$$

$$\tilde{W}(x_F, b_T, Q) = \sum_j H^{\rm DY}_{j\bar{j}}(Q, \mu, a_s(\mu))\tilde{f}_{j/A}(x_A, b_T; \zeta_A, \mu)\tilde{f}_{\bar{j}/B}(x_B, b_T; \zeta_B, \mu)$$

Renormalization and TMD Evolution- $\{\zeta, \mu\}$

- Collins Soper Eq. $$\frac{\partial \ln \tilde{f}_{j/H}(x, b_T; \mu, \zeta)}{\partial \ln \sqrt{\zeta}} = \tilde{K}(b_T, \mu)$$

$$\tilde{K}(b_T, \mu) \equiv \frac{1}{2} \frac{\partial}{\partial y_n} \ln \frac{S(b_T, y_n, -\infty)}{S(b_T, y_n, -\infty)}$$

- RGE for C.S. kernel

$$\frac{d\tilde{K}(b_T; \mu)}{d \ln \mu} = -\gamma_k(\alpha_s(\mu))$$

- RGE for TMD $$\frac{d \ln \tilde{f}_{j/H}(x, b_T; \mu, \zeta)}{d \ln \mu} = -\gamma_F(\alpha_s(\mu), \zeta/\mu)$$

Solve simultaneously and get evolved renormalized TMD $\rightarrow \zeta = Q^2, \quad \mu = \mu_Q \sim Q$

TMD factorization

W – term

- In small-p_T region, Use the CSS formalism for TMD evolution

$$\frac{d\sigma}{dQ^2\,dy\,dq_T^2} = \frac{4\pi^2\alpha^2}{9Q^2 s}\sum_{j,j_A,j_B} H^{DY}_{j\bar{j}}(Q,\mu_Q,a_s(\mu_Q))\int\frac{d^2\boldsymbol{b}_T}{(2\pi)^2}e^{i\boldsymbol{q}_T\cdot\boldsymbol{b}_T}$$
$$\times e^{-g_{j/A}(x_A,b_T;b_{max})}\int_{x_A}^1\frac{d\xi_A}{\xi_A} f_{j_A/A}(\xi_A;\mu_{b_*})\tilde{C}^{PDF}_{j/j_A}\left(\frac{x_A}{\xi_A},b_*;\mu_{b_*}^2,\mu_{b_*},a_s(\mu_{b_*})\right)$$
$$\times e^{-g_{\bar{j}/B}(x_B,b_T;b_{max})}\int_{x_B}^1\frac{d\xi_B}{\xi_B} f_{j_B/B}(\xi_B;\mu_{b_*})\tilde{C}^{PDF}_{\bar{j}/j_B}\left(\frac{x_B}{\xi_B},b_*;\mu_{b_*}^2,\mu_{b_*},a_s(\mu_{b_*})\right)$$
$$\times\exp\left\{-g_K(b_T;b_{max})\ln\frac{Q^2}{Q_0^2}+\tilde{K}(b_*;\mu_{b_*})\ln\frac{Q^2}{\mu_{b_*}^2}+\int_{\mu_{b_*}}^{\mu_Q}\frac{d\mu'}{\mu'}\left[2\gamma_j(a_s(\mu'))-\ln\frac{Q^2}{(\mu')^2}\gamma_K(a_s(\mu'))\right]\right\}$$

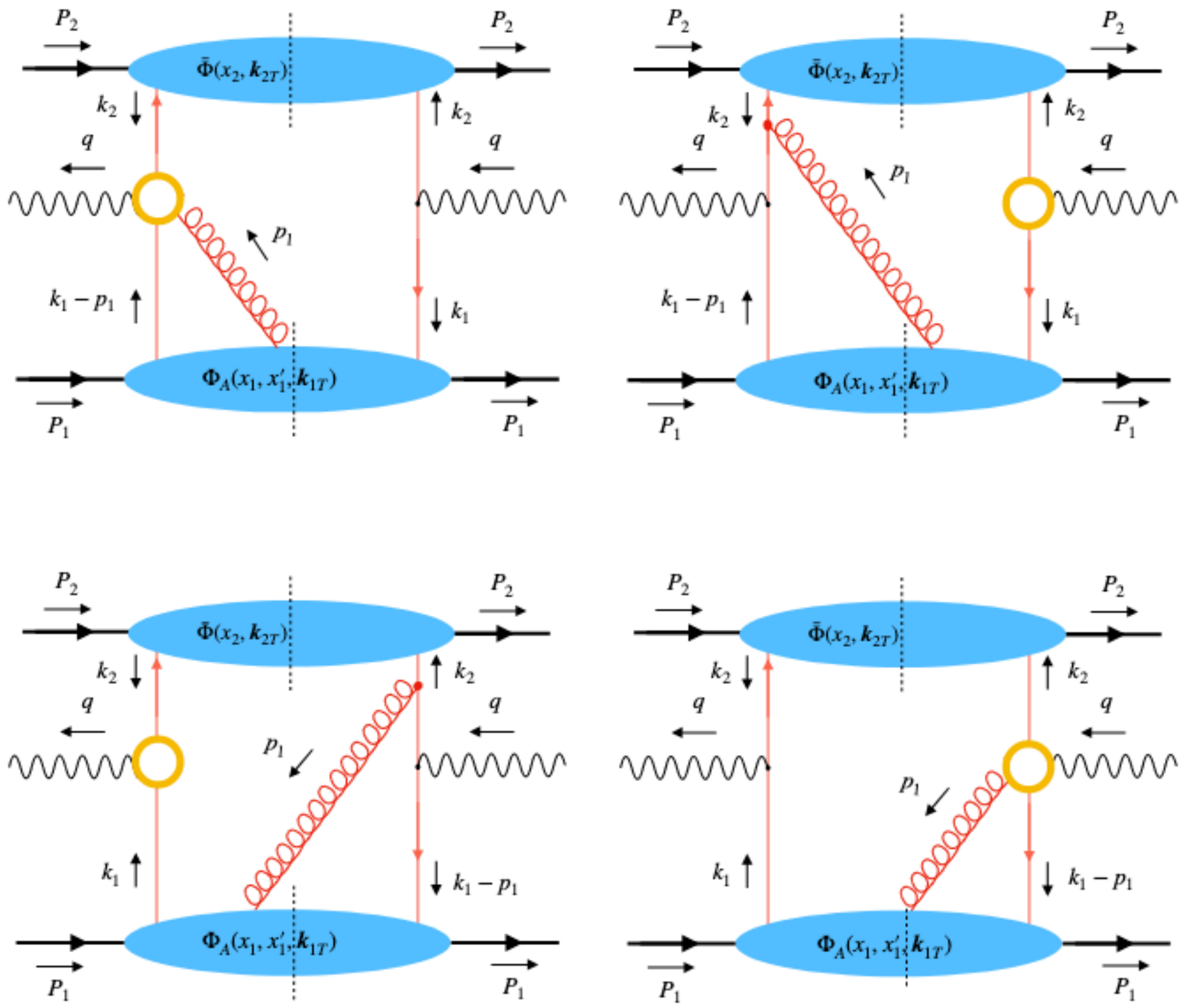
Non-perturbative TMDs to extract

TMD Factorization

- ✦ *Collins Soper Sterman NPB 1985*
- ✦ *Ji Ma Yuan PRD PLB …2004, 2005*
- ✦ *Aybat Rogers PRD 2011*
- ✦ *Collins 2011 Cambridge Press*
- ✦ *Echevarria, Idilbi, Scimemi JHEP 2012, …*
- ✦ *SCET Becher & Neubert, 2011 EJPC*

- Perturbative content calculated from first principles of QFT
- Non-perturbative Collinear pdfs &
- TMD to be fit to data

Organize via Fierz decomp motivate TMD factorization framework



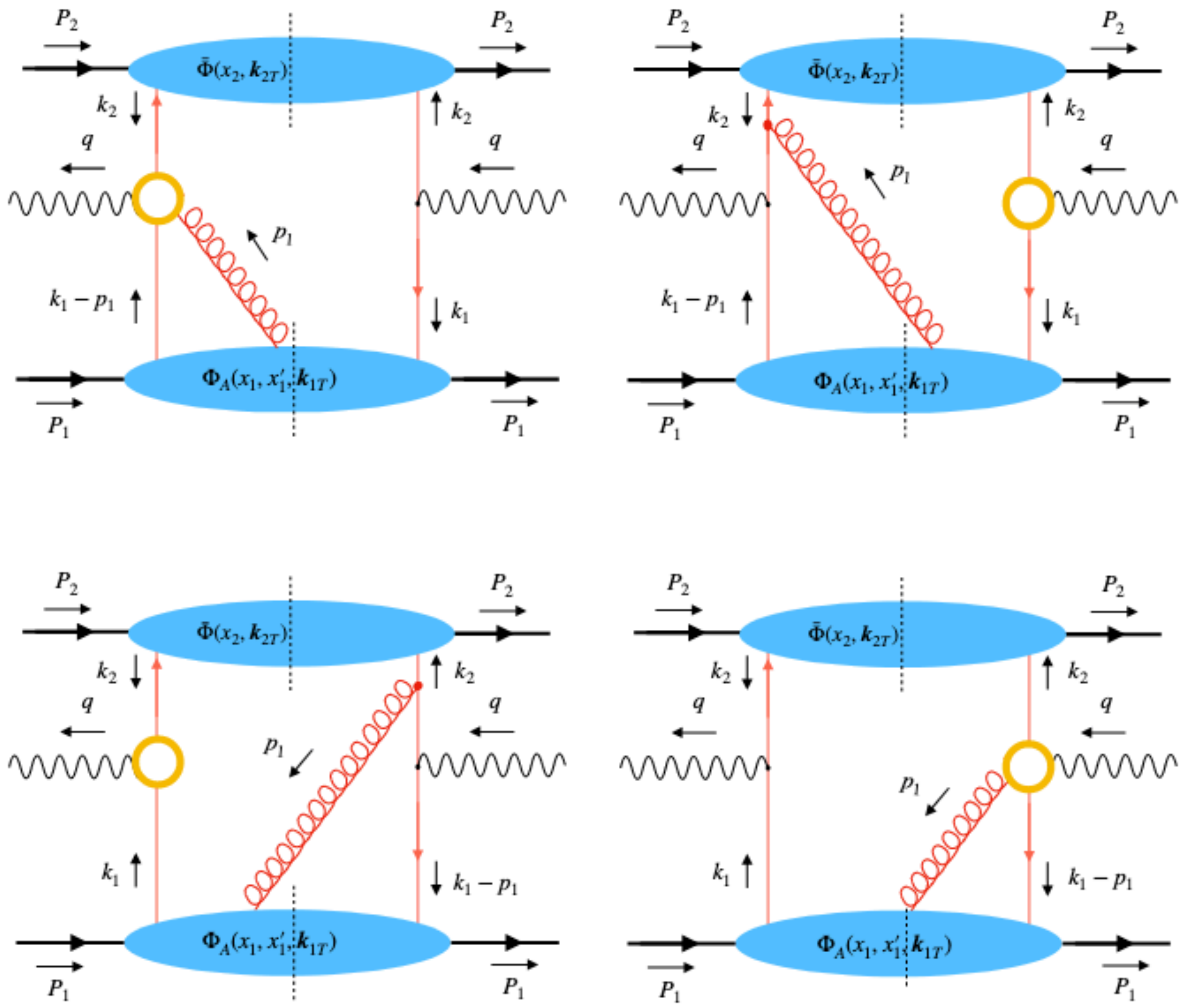
Twist 2	Twist 3	Twist 4
$\frac{1}{2}\slashed{n}$, $\frac{1}{4}\slashed{\bar{n}}$	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}\slashed{\bar{n}}$, $\frac{1}{4}\slashed{n}$
$\frac{1}{2}\slashed{n}\gamma^5$, $\frac{1}{4}\gamma^5\slashed{\bar{n}}$	$\frac{1}{2}\gamma^5$, $\frac{1}{2}\gamma^5$	$\frac{1}{2}\slashed{\bar{n}}\gamma^5$, $\frac{1}{4}\gamma^5\slashed{n}$
$\frac{i}{2}\sigma^{k+}\gamma^5$, $\frac{i}{4}\gamma^5\sigma_{-k}$	$\frac{1}{2}\gamma^k$, $\frac{1}{2}\gamma_k$	$\frac{i}{2}\sigma^{k-}\gamma^5$, $\frac{i}{4}\gamma^5\sigma_{+k}$
	$\frac{1}{2}\gamma^k\gamma^5$, $\frac{1}{2}\gamma^5\gamma_k$	
	$\frac{i}{2}\sigma^{kl}\gamma^5$, $\frac{i}{4}\gamma^5\sigma_{lk}$	
	$\frac{i}{4}\sigma^{+-}\gamma^5$, $\frac{i}{4}\gamma^5\sigma_{+-}$	

$$\mathrm{Tr}\left[\gamma^\mu\,\bar{\Gamma}_1^{a1}\,\gamma^\nu\,\bar{\Gamma}_1^{a2}\right] \to \mathrm{Tr}\left[\gamma^\mu\,\left(1+F(Q;\mu)\right)\bar{\Gamma}_1^{a1}\,\gamma^\nu\,\left(1+F(Q;\mu)\right)\bar{\Gamma}_1^{a2}\right]$$

$$W^{\mu\nu}_{\{2,3\,\mathrm{intrinsic}\}} = \frac{1}{N_c}\sum_{a1,a2}\sum_q e_q^2 \int d^2\boldsymbol{k}_{1T}\,d^2\boldsymbol{k}_{2T}\,\delta^{(2)}\left(\boldsymbol{q}_T-\boldsymbol{k}_{1T}-\boldsymbol{k}_{2T}-\boldsymbol{\lambda}_T\right)\,S(\boldsymbol{\lambda}_T;\mu,\nu)$$

$$\times\,\mathrm{Tr}\left[\gamma^\mu\,\bar{\Gamma}_1^{a1}\,\gamma^\nu\,\bar{\Gamma}_1^{a2}\right]\,\Phi^{[\Gamma^{a1}]}\left(x_1,\boldsymbol{k}_{1T},\boldsymbol{S}_1;\mu,\zeta_1/\nu^2\right)\,\bar{\Phi}^{[\Gamma^{a2}]}\left(x_2,\boldsymbol{k}_{2T},\boldsymbol{S}_2;\mu,\zeta_2/\nu^2\right)\,.$$

Organize via Fierz decomp motivate TMD factorization framework



Twist 2	Twist 3	Twist 4
$\frac{1}{2}\not{n}\ ,\ \frac{1}{4}\not{\bar{n}}$	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{2}\not{\bar{n}}\ ,\ \frac{1}{4}\not{n}$
$\frac{1}{2}\not{n}\gamma^5,\ \frac{1}{4}\gamma^5\not{\bar{n}}$	$\frac{1}{2}\gamma^5,\ \frac{1}{2}\gamma^5$	$\not{\bar{n}}\gamma^5,\ \frac{1}{4}\gamma^5\not{n}$
$\frac{i}{2}\sigma^{k+}\gamma^5,\ \frac{i}{4}\gamma^5\sigma_{-k}$	$\frac{1}{2}\gamma^k,\ \frac{1}{2}\gamma_k$	$\frac{i}{2}\sigma^{k-}\gamma^5\ ,\ \frac{i}{4}\gamma^5\sigma_{+k}$
	$\frac{1}{2}\gamma^k\gamma^5,\ \frac{1}{2}\gamma^5\gamma_k$	
	$\frac{i}{2}\sigma^{kl}\gamma^5,\ \frac{i}{4}\gamma^5\sigma_{lk}$	
	$\frac{i}{4}\sigma^{+-}\gamma^5,\ \frac{i}{4}\gamma^5\sigma_{+-}$	

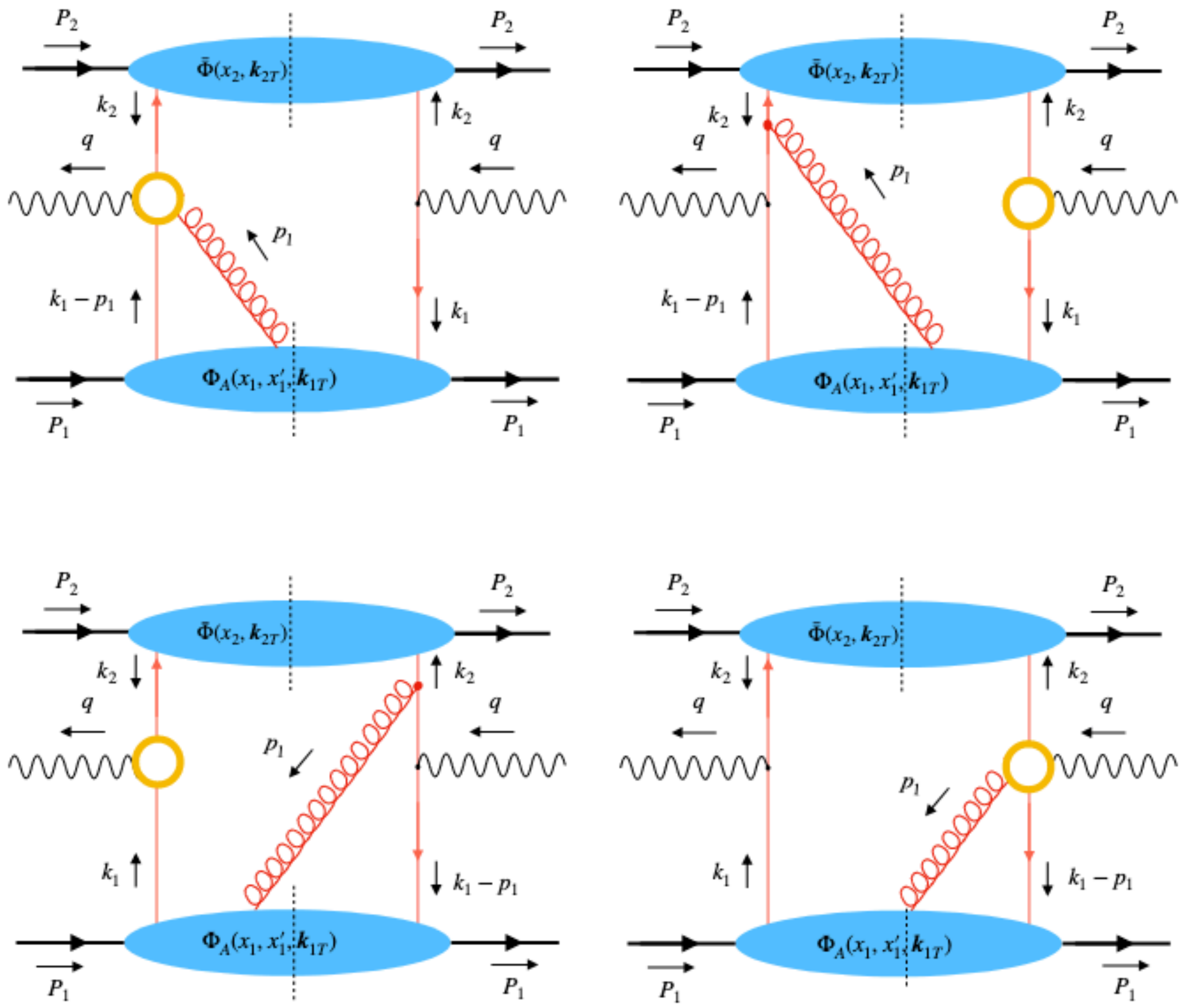
$$\mathrm{Tr}\left[\gamma^\mu\,\bar{\Gamma}_1^{a1}\,\gamma^\nu\,\bar{\Gamma}_1^{a2}\right] \to \mathrm{Tr}\left[\gamma^\mu\,\left(1+F(Q;\mu)\right)\bar{\Gamma}_1^{a1}\,\gamma^\nu\,\left(1+F(Q;\mu)\right)\bar{\Gamma}_1^{a2}\right]$$

$$W_3^{\mu\nu} = \frac{1}{N_c}\sum_q e_q^2 \int d^2\boldsymbol{k}_{1T}\, d^2\boldsymbol{k}_{2T}\, d^2\boldsymbol{\lambda}_T \delta^{(2)}(\boldsymbol{k}_{1T}+\boldsymbol{k}_{2T}+\boldsymbol{\lambda}_T-\boldsymbol{q}_T) S'(\boldsymbol{\lambda}_T)$$

$$\times\left\{\int dx_1'\, \mathrm{Tr}\left[\gamma_\rho \frac{\gamma^-}{\sqrt{2}Q}\Gamma_3^\nu(Q;\mu)\Phi_F^\rho(x_1,x_1',\boldsymbol{k}_{1T})\Gamma_3^\mu(Q;\mu)\bar{\Phi}(x_2,\boldsymbol{k}_{2T})\right]\right.$$

$$\left. + \begin{matrix}(k_1 \leftrightarrow k_2)\\ (\mu \leftrightarrow \nu)^*\end{matrix}\right\},$$

Organize via Fierz decomp motivate TMD factorization framework



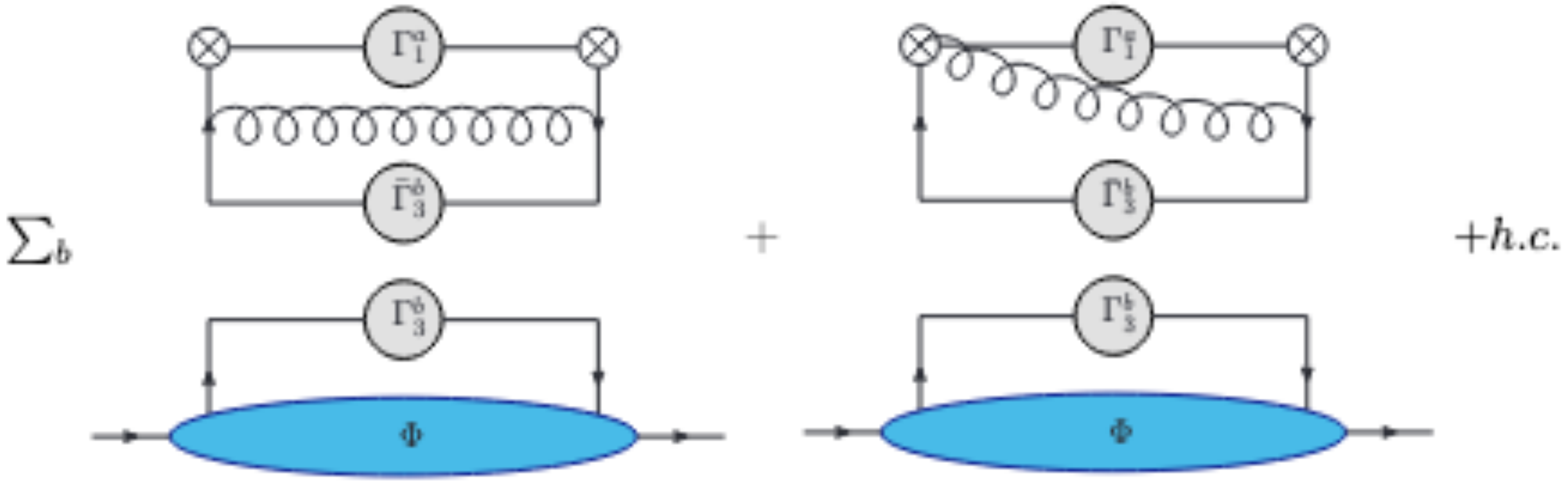
Twist 2	Twist 3	Twist 4
$\frac{1}{2}\not{n}\ ,\ \frac{1}{4}\not{\bar{n}}$	$\frac{1}{2},\frac{1}{2}$	$\frac{1}{2}\not{\bar{n}}\ ,\ \frac{1}{4}\not{n}$
$\frac{1}{2}\not{n}\gamma^5,\ \frac{1}{4}\gamma^5\not{\bar{n}}$	$\frac{1}{2}\gamma^5,\ \frac{1}{2}\gamma^5$	$\frac{1}{2}\not{\bar{n}}\gamma^5,\ \frac{1}{4}\gamma^5\not{n}$
$\frac{i}{2}\sigma^{k+}\gamma^5,\ \frac{i}{4}\gamma^5\sigma_{-k}$	$\frac{1}{2}\gamma^k,\ \frac{1}{2}\gamma_k$	$\frac{i}{2}\sigma^{k-}\gamma^5\ ,\ \frac{i}{4}\gamma^5\sigma_{+k}$
	$\frac{1}{2}\gamma^k\gamma^5,\ \frac{1}{2}\gamma^5\gamma_k$	
	$\frac{i}{2}\sigma^{kl}\gamma^5,\ \frac{i}{4}\gamma^5\sigma_{lk}$	
	$\frac{i}{4}\sigma^{+-}\gamma^5,\ \frac{i}{4}\gamma^5\sigma_{+-}$	

$$L_{\mu\nu}\,\mathrm{Tr}\left[\gamma^\mu\,\bar{\Gamma}_1^{a1}\,\gamma^\nu\,\bar{\Gamma}_1^{a2}\right] = Q^2 f\,(\theta,\phi)\;H(Q;\mu) + \mathcal{O}(\alpha_s^2)$$

Only virtual graphs enter Due to this relation, higher loop expression for the vertex leave the Lorentz structure of the trace unchanged in the TMD region.

$$\frac{d\sigma}{d^4q\,d\Omega} = \frac{\alpha_{\rm em}^2}{2sQ^2}\left(\frac{1+\cos^2\theta}{2}\right)H(Q;\mu)\sum_q e_q^2\int d^2\boldsymbol{k}_{1T}\,d^2\boldsymbol{k}_{2T}\,d^2\boldsymbol{\lambda}_T\,\delta^{(2)}(\boldsymbol{k}_{1T}+\boldsymbol{k}_{2T}+\boldsymbol{\lambda}_T-\boldsymbol{q}_T)$$
$$\times\, f_1\left(x_1,\boldsymbol{k}_{1T};\mu,\zeta_1/\nu^2\right)\;f_1\left(x_2,\boldsymbol{k}_{2T};\mu,\zeta_2/\nu^2\right)\,S(\boldsymbol{\lambda}_T;\mu,\nu) \qquad (4.5)$$

Return to matching need subtracted TMDs in b-space



$$f_1^{(1)}(x_1, b^2) = \frac{1}{M_1^2 b^2} \frac{\alpha_s}{2\pi^2} \left[\frac{1}{2} L\left(\frac{Q^2}{\mu_b^2}\right) f_1(x_1, Q^2) + \left(P_{qq} \otimes f_1\right)(x_1, Q^2) \right] ,$$

$$f^{\perp(1)}(x_1, b^2) = \frac{1}{M_1^2 b^2} \frac{\alpha_s}{4\pi^2} \left[\frac{1}{2} L\left(\frac{Q^2}{\mu_b^2}\right) f_1(x_1, Q^2) + \left(P'_{qq} \otimes f_1\right)(x_1, Q^2) \right] ,$$

where the two splitting kernels $P_{qq}(x)$ and $P'_{qq}(x)$ are consistent with [26] and provided below

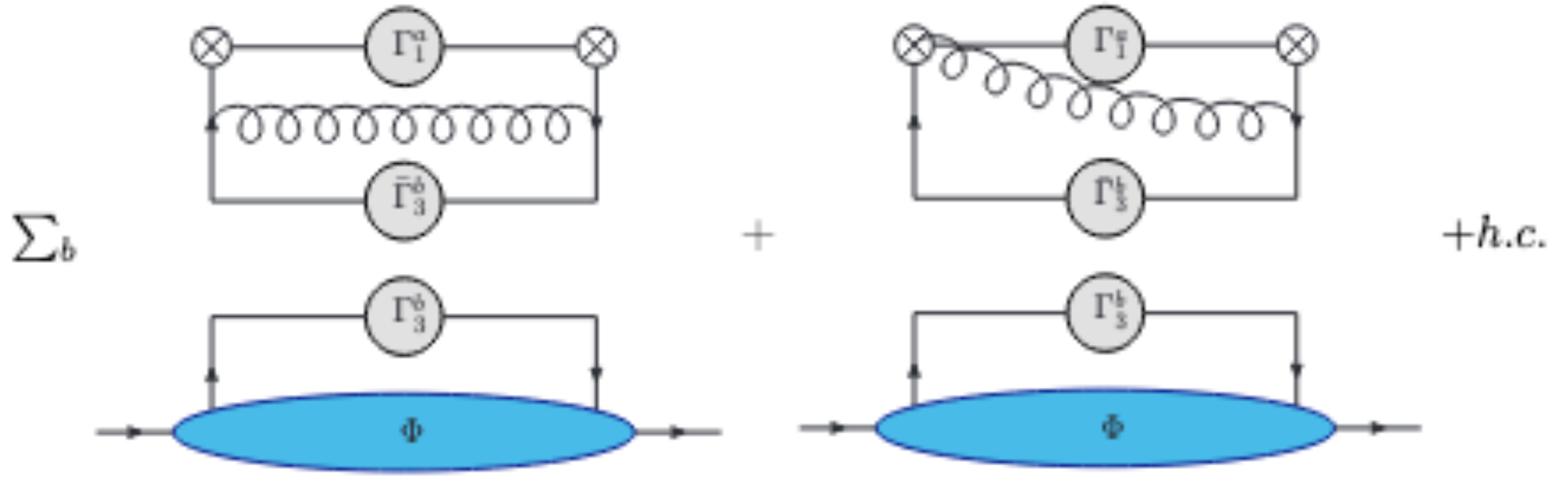
$$P_{qq}(x) = \left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x) \right] , \quad P'_{qq}(x) = \left[\frac{2x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x) \right] . \qquad (4.63)$$

Return to matching need subtracted TMDs in b-space

Bacchetta et al. PLB 2019 state that including the ("square root of the") soft factor in the definition of TMDs removes the rapidity divergences, which in practice reduces to the a series of replacements w.r.t. the formulas in Bacchetta et. al. JHEP 2008 matches and mismatches paper

At present we are finding that for the intrinsic function

Return to matching

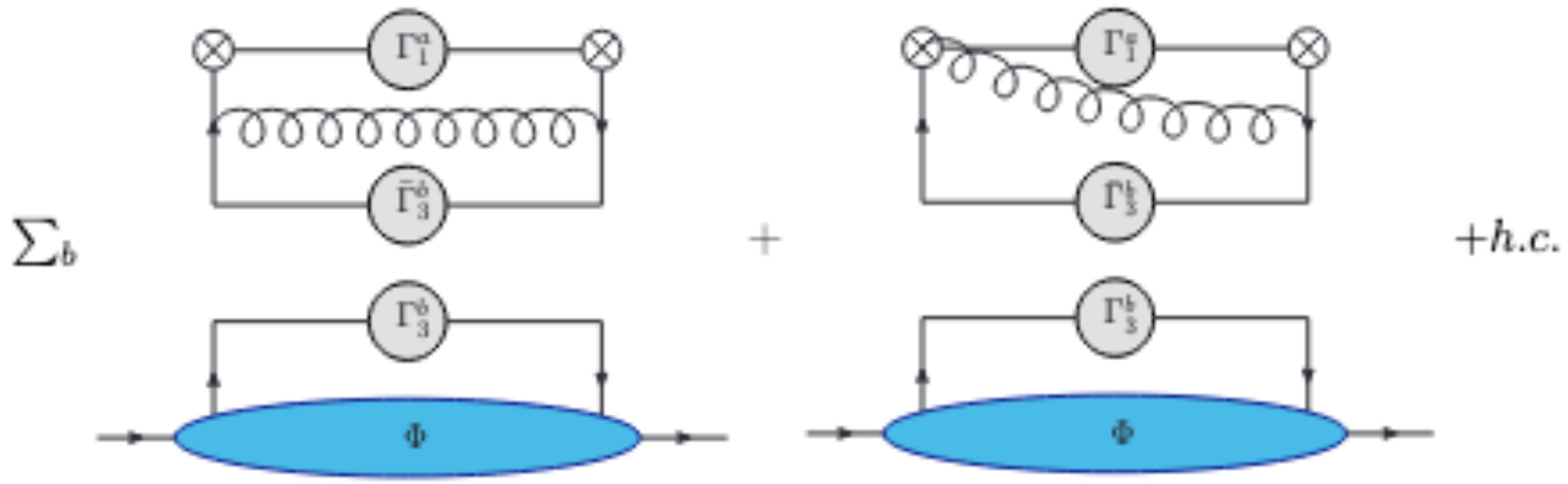


Asymptotic term for cos ϕ
Obtained by performing a "large kT" /matched TMDs
Into Bessel Weighted structure function ie perform b-space Fourier transform integral

$$F_{UU}^{\cos\phi} = \frac{2}{Q}\sum_q e_q^2 \int dbb^2 J_1(bq_T)\left\{\frac{\alpha_s}{b^2 4\pi^2}\left[\quad \ldots\ldots \right.\right.$$

$$\begin{aligned} F_{UU}^{\cos\phi_h} = -\frac{1}{Qq_T}\frac{\alpha_s}{2\pi^2 z^2}\Bigg\{ \sum_a x e_a^2 \Bigg[L\left(\frac{Q^2}{q_T^2}\right) f_1^a(x,Q^2)\, D_1^a(z,Q^2) \\ + \sum_{i=a,g}\Big(f_1^a(x;Q^2)(D_1^i \otimes P'_{ia})(z;Q^2) + (P'_{ai}\otimes f_1^i)(x;Q^2)\, D_1^a(z;Q^2)\Big)\Bigg] \\ + \mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{q_T}\right) + \mathcal{O}\left(\frac{q_T}{Q}\right)\Bigg\}. \end{aligned}$$

Return to matching



Asymptotic term for cos ϕ
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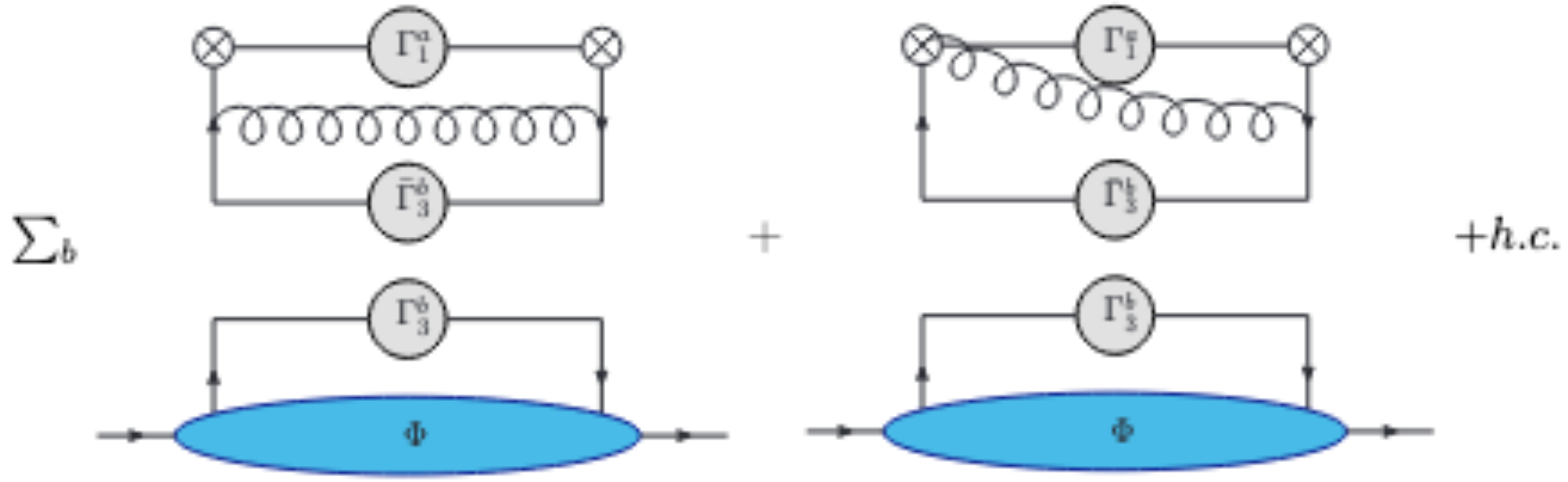
$$F_{UU}^{\cos\phi_h} = -\frac{1}{Qq_T}\frac{\alpha_s}{2\pi^2 z^2}\Bigg\{\sum_a x e_a^2 \Bigg[L\left(\frac{Q^2}{q_T^2}\right) f_1^a(x,Q^2)\, D_1^a(z,Q^2)$$
$$+\sum_{i=a,g}\Big(f_1^a(x;Q^2)(D_1^i \otimes P'_{ia})(z;Q^2) + (P'_{ai}\otimes f_1^i)(x;Q^2)\, D_1^a(z;Q^2)\Big)\Bigg]$$
$$+\mathcal{O}\left(\frac{\Lambda_{\rm QCD}}{q_T}\right) + \mathcal{O}\left(\frac{q_T}{Q}\right)\Bigg\}.$$

Agrees with Mendez NPB 1978 after determining the AY term intermediate region $\Lambda_{QCD} \ll q_T \ll Q$, from the collinear factorized

Look under the hood rapidity subtraction



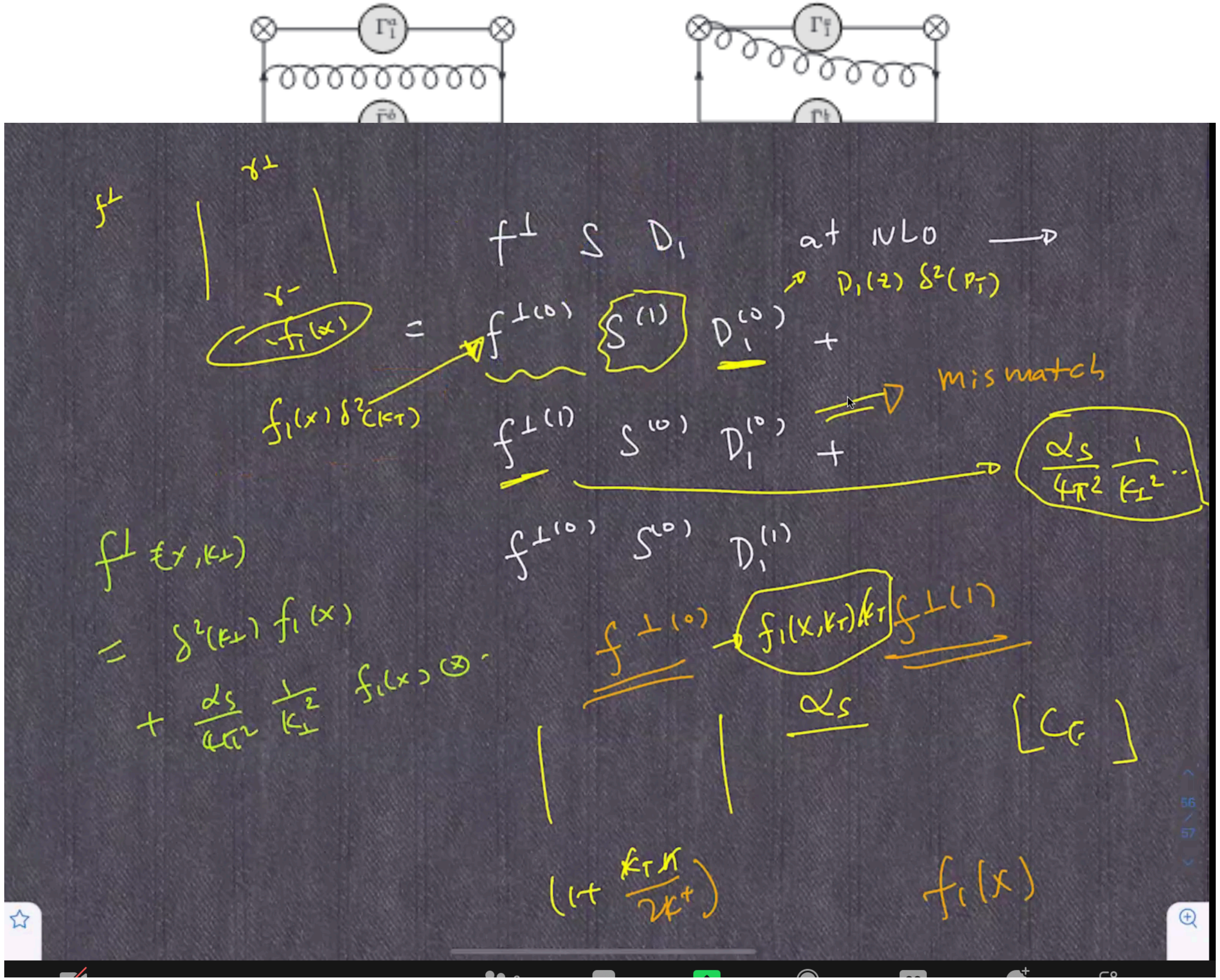
$$\sigma(q_T) \sim f^{\perp}(k_T)S(\lambda)D_1(p_T)$$

$$= f^{\perp(0)}(k_T)S^{(1)}(\lambda)D_1^{(0)}(p_T)$$

$$+ f^{\perp(1)}(k_T)S^{(0)(\lambda)}D_1^{(0)}(p_T)$$

$$+ f^{\perp(0)}(k_T)S^{(0)}(\lambda)D_1^{(1)}(p_T)$$

Return to matching



Summary

Bacchetta et al. PLB 2019 state that including the ("square root of the") soft factor in the definition of TMDs removes the rapidity divergences, which in practice reduces to the a series of replacements w.r.t. the formulas in Bacchetta et. al. JHEP 2008 matches and mismatches paper

At present we are finding that for the intrinsic function

$$f^{\perp}(x,\boldsymbol{k}_T) = \delta^{(2)}(\boldsymbol{k}_\perp) f_1(x) + \frac{\alpha_s C_F}{4\pi^2 k_T^2}\Big(-\frac{2}{\eta}\delta(1-x) + P'_{qq}(x) - \frac{3}{2}\delta(1-x) - \ln\left(\frac{\nu^2}{P^{+2}}\right)\delta(1-x)\Big) \otimes f_1(x)\,, \quad \text{(A.10)}$$

$$S_q(\boldsymbol{\lambda}_\perp,\mu,\nu) = \delta^{(2)}(\boldsymbol{\lambda}_\perp) + \frac{\alpha_s C_F}{\pi^2 \lambda_\perp^2}\left(\frac{2}{\eta} + \ln\left(\frac{\nu^2}{\lambda_\perp^2}\right)\right)\,, \quad \text{(A.11)}$$

$$\hat{f}^{\perp}(x,\boldsymbol{k}_T) = f^{\perp}(x,\boldsymbol{k}_T) \otimes \sqrt[4]{S_q} = f_1(x)\delta^{(2)}(\boldsymbol{k}_\perp) + \frac{\alpha_s C_F}{4\pi^2 k_T^2}\left[\frac{1}{2}L\left(\frac{P^{+2}}{k_T^2}\right) f_1(x) + P'_{qq}\otimes f_1(x)\right]\,, \quad \text{(A.12)}$$

Summary

$f^\perp$ intrinsic: trace with γ^i

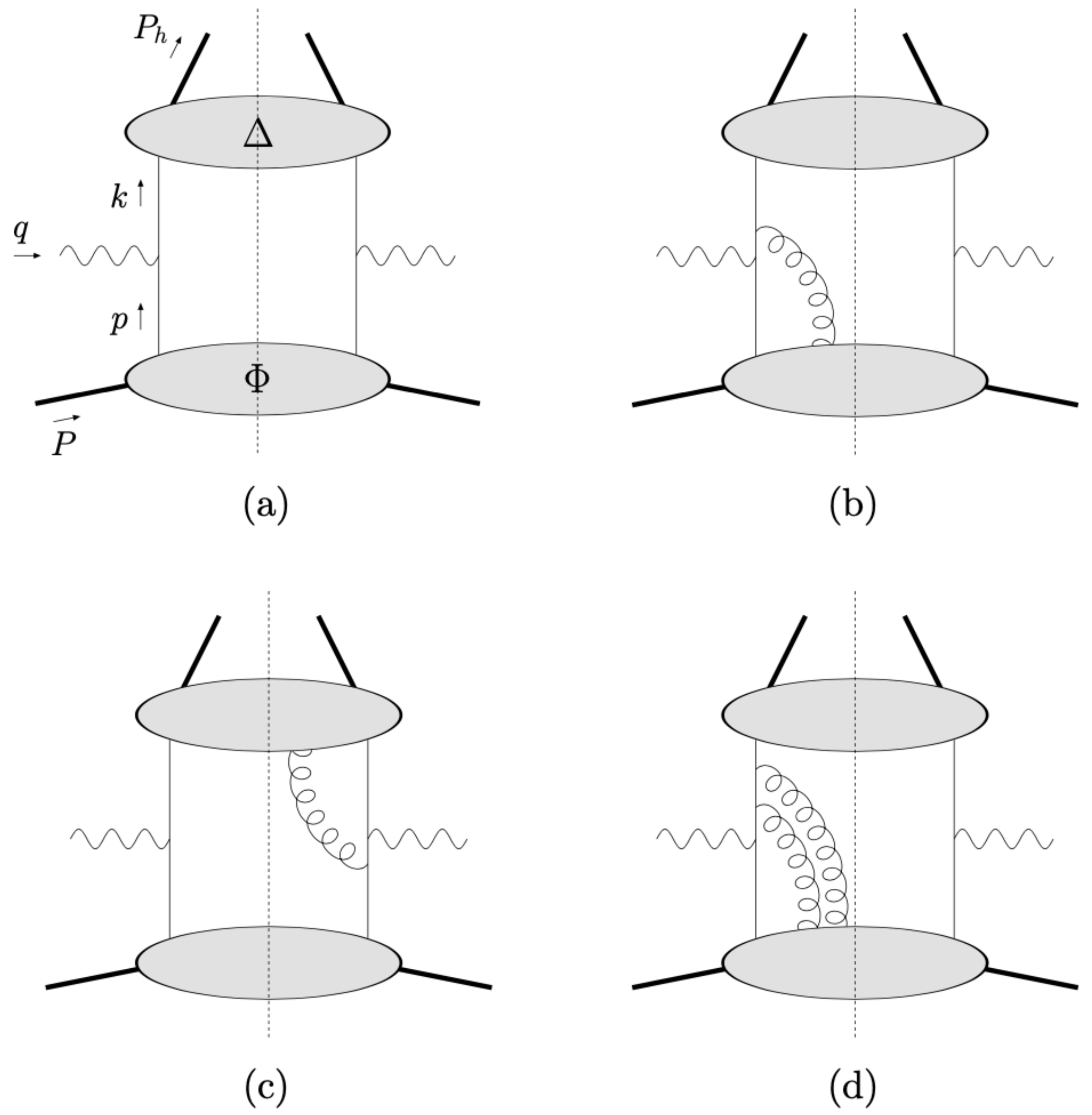
$$\Phi_{jj'}(x, \boldsymbol{k}_T, \boldsymbol{S}; \mu, \nu) = \int \frac{d\xi^- d^2\xi_T}{(2\pi)^3}\, e^{ik\cdot\xi} \tag{2.20}$$

$$\times \left\langle P, \boldsymbol{S} \middle| \bar{\psi}_{j'}(\xi)\, \mathcal{U}^{\bar{n}}_{(\xi^-,-\infty;\boldsymbol{\xi}_T)}\, \mathcal{U}^{T}_{(\boldsymbol{\xi}_T,\boldsymbol{0};-\infty)}\, \mathcal{U}^{\bar{n}}_{(-\infty,0,\boldsymbol{0}_T)}\, \psi_j(0) \middle| P, \boldsymbol{S} \right\rangle \Big|_{\xi^+=0} .$$

Here mixes good and bad light cone components of fields.
Ebert Gao Stewart twist 3 factorization in SCET in terms of good light cone functions in correlators.

Extras

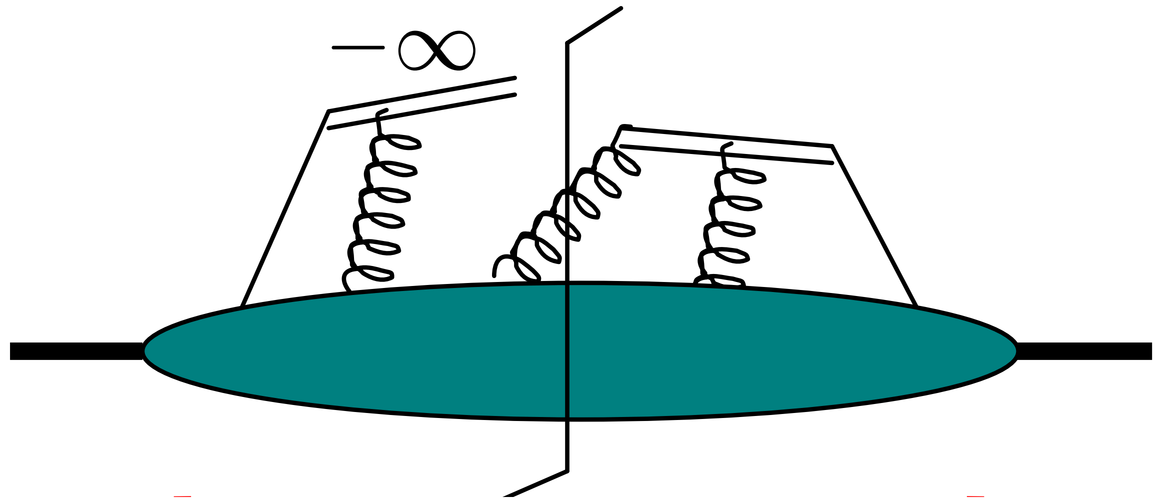
Tree level factorization leading power



SIDIS tree-level diagrams relevant for leading-power observables. Here unsuppressed gluon $A^+ = n_- \cdot A$ exchange factorize into gauge links rendering gauge invariant "twist-2" TMDs

- ✦Mulders Tangerman NPB1995
- ✦Boer Mulders PRD 1998
- ✦Collins PLB 2002, Ji Yuan PLB 2002
- ✦Goeke Metz Schlegel PLB 2005
- ✦Bacchetta et al 2007 JHEP

$$\Phi(x, \boldsymbol{k}_T)$$



$$2M\mathcal{W}^{\mu\nu} = e^2 \int d^2\boldsymbol{p}_T d^2\boldsymbol{k}_T \delta^2(\boldsymbol{p}_T + \boldsymbol{q}_T - \boldsymbol{k}_T)\left\{\mathrm{Tr}\left[\Phi(p)\gamma_\mu\Delta(k)\gamma_\nu\right]\right\}$$

Gauge invariant correlators

Tree level factorization leading power

- ✦Mulders Tangerman NPB1995
- ✦Boer Mulders PRD 1998
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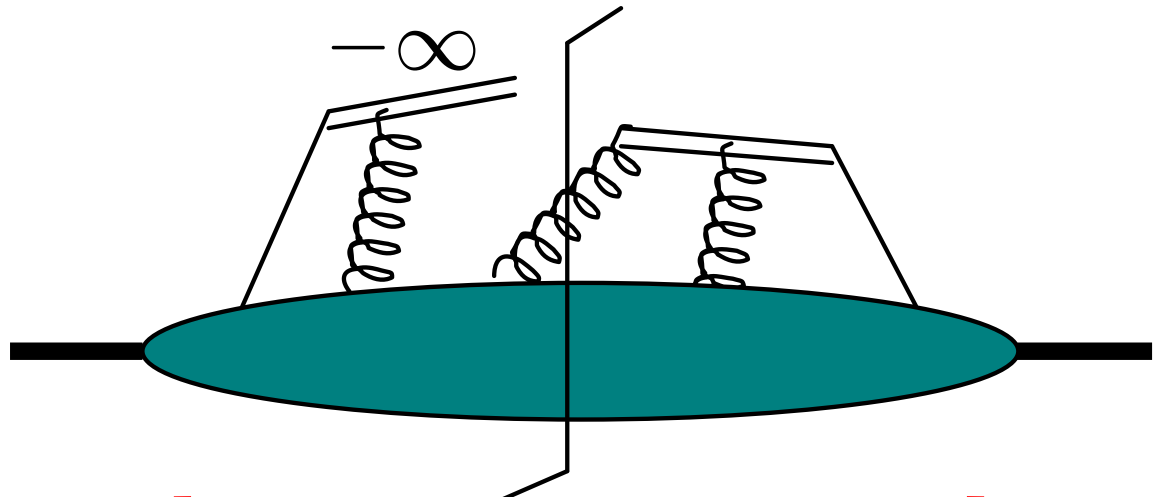
- ✦Bacchetta et al 2007 JHEP

SIDIS tree-level diagrams relevant for leading-power observables. Here unsuppressed gluon $A^+ = n_- \cdot A$ exchange factorize into gauge links rendering gauge invariant “twist-2” TMDs

Leading Quark TMDPDFs (Nucleon Spin, Quark Spin)

Nucleon Polarization	Quark Polarization: Un-Polarized (U)	Longitudinally Polarized (L)	Transversely Polarized (T)
U	f_1 = Unpolarized		$h_1^\perp$ = Boer-Mulders
L		g_1 = Helicity	$h_{1L}^\perp$ = Worm-gear
T	$f_{1T}^\perp$ = Sivers	$g_{1T}^\perp$ = Worm-gear	h_1 = Transversity; $h_{1T}^\perp$ = Pretzelosity

$$\Phi(x, \boldsymbol{k}_T)$$



$$\Phi^{(2)}(x, \boldsymbol{k}_T, \boldsymbol{S}) = \left(f_1 - \frac{\epsilon_T^{\rho\sigma} k_{T\rho} S_{T\sigma}}{M} f_{1T}^\perp\right) \frac{\not{n}_+}{4} + \left(\lambda g_{1L} - \frac{\boldsymbol{k}_T \cdot \boldsymbol{S}_T}{M} g_{1T}\right) \frac{\gamma^5 \not{n}_+}{4}$$

$$+ \left(S_T^k h_1 + \frac{\lambda k_T^k}{M} h_{1L}^\perp - \frac{\epsilon_T^{kj} k_{Tj}}{M} h_1^\perp - \frac{k_T^k k_T^j - \frac{1}{2} k_T^2 g_T^{kj}}{M^2} S_{Tj} h_{1T}^\perp\right) \frac{i\gamma^5 \sigma_{-k}}{4}$$