Prospects for Lattice Supersymmetry and Chiral Fermions

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Overview:

Lattice gauge theory and USQCD are poised to have significant impact on long standing problems in theoretical physics

- Holographic approaches to quantum gravity. Formulation (and parallelized code) available for $\mathcal{N} = 4$ SYM - capable of probing quantum gravity away from large N_c and weak coupling.
- Recent work in condensed matter physics – symmetric mass generation (SMG) – opens up new possibilities for constructing lattice models targeting chiral fermions in the continuum limit.

Not comprehensive review. Reflects personal interests. There are other things out there eg DWF/overlap approaches to SUSY and chiral fermions

$\mathcal{N} = 4$ SUSY on lattice

Key features:

- Fermions – Kähler-Dirac (KD) fields. Can discretize KD fermions and obtain staggered fermions BUT DON'T!. Instead place fermions on links ... no (unwanted) doubling !
- Bosons: Complex gauge field (to accommodate the scalars). Scalar supersymmetry: $\mathcal{Q}\mathcal{U}_\mu(x) = \psi_\mu(x)$. Compatible with lattice gauge invariance.
- $\mathcal{Q}^2 = 0$. Lattice action $S = \mathcal{Q}\,(\text{something})$.
- Optimized parallel code (based on MILC libraries) exists (David Schaich). Recent work (JHEP 12 (2020) 140) with modified action allows simulations at strong coupling $\lambda \sim 10 - 100$
- Sign problem observed to be negligible .. no proof (yet) ? ..

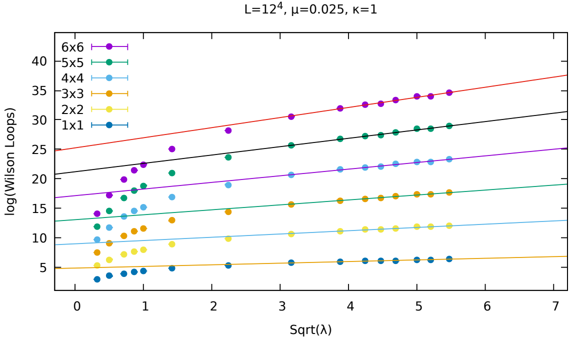
Applications: Wilson loops

Maldacena tell us that (supersymmetric) Wilson loops ($N_c, \lambda \to \infty$):

$$W(R,T) \sim e^{-\frac{c\sqrt{\lambda}}{R}T}$$

No confinement: non-abelian Coulomb form.

Note: $\sqrt{\lambda} \sim g$ – non-perturbative. Property of dual gravity



To do ..

- Detailed studies of static quark potential for $N = 2, 3$ on larger lattices. Planar result seems to appear even for $N = 2$ - why ? (string loops ..)
- Conformal phase: anomalous dimensions of non-BPS ops eg. Konishi operator $\sum_A \mathrm{Tr} X_A^2$ at strong coupling.
- $\mathcal{N} = 4$ has Higgs phases where $< X > \neq 0$. Good place to study S-duality – $g \rightarrow \frac{1}{g}$. Exchanges (massive) W boson with monopole $\leftarrow$ twisted boundary conditions.
- At finite temperature can compute thermodynamics of black holes in dual gravity theory. Simulate theory in
  - $D = 3$ (PRD102 (2020) 106009, only USQCD)
  - $D = 2$ (PRD97 (2018) 086020, only USQCD)
  - $D = 1$ (eg PRD 94 (2016) 094501, many groups ..).

Bigger question: how can we use simulations of $\mathcal{N} = 4$ to probe the nature of the dual quantum gravity theory ?

Lattice chiral fermions

Very little is known about strongly coupled chiral gauge theories

Many efforts to construct lattice chiral gauge theories. Wilson, staggered, DWF and overlap formulations. No success ..

- Nielsen-Ninomiya forces one to start with Dirac fermions.
- Two main approaches:
  - Separate L and R modes in 5th dimension. Decoupling tricky with dynamical gauge fields.
  - Design bulk interactions to lift say R modes to the cut-off - mirror models. Use four fermion operators to gap. Typically symmetry breaking fermion condensates form recoupling L and R.

Symmetric Mass Generation - part 1

Can we gap fermions without breaking symmetries ?
In principle yes – provided we cancel off all anomalies

Consider $SU(5)$ gauge theory with global $G = U(1)$
L Weyl fields: $\chi_{\alpha\beta}(1)$ and $\psi^{\alpha}(-3)$

No mass term possible but gauge anomaly cancels $A(\bar{\mathbf{5}}) = -\mathbf{A}(\mathbf{10})$

But can imagine weakly gauging G – 't Hooft anomaly:
$$\sum_a Q_a^3 = 5 \times (-3)^3 + 10 \times (1)^3 = -125$$

Key: 't Hooft anomalies must be same in IR and UV
Options in IR:

- Composite gauge singlet massless fermions
- Goldstone bosons from breaking $G \leftarrow$ avoid ?

Symmetric mass generation - part 2

Expect color singlet composites in I.R. One obvious candidate:
$\overline{\xi} = \chi_{ab}\psi^a\psi^b$ – $U(1)$ charge -5
Precisely what is needed for the anomaly $(-5)^3 = -125$!
Can satisfy 't Hooft anomaly if $\overline{\xi}$ remains massless!.

A small twist

If all states are to be massive in I.R must cancel anomaly in U.V
Just add a singlet $\xi(+5)$!
Can now gap $\overline{\xi}$ with Dirac mass term
$G\overline{\xi}\xi = G\chi_{ab}\psi^a\psi^b\xi \leftarrow \text{four fermion term}$
Preserves G - symmetric mass generation SMG

Another observation

Notice for SMG sixteen Weyl fermions needed.

Discrete anomalies

In fact 16 fermions necessary to cancel off a new discrete anomaly:

$$\psi_L \to -i\psi_L \quad \psi_R \to +i\psi_R \quad \leftarrow \text{spin-}Z_4 \text{ symmetry}$$

Anomaly cancels if

$$n_L - n_R = 0 \mod 16$$

To gap states in IR without breaking symmetries must ensure that all anomalies cancel in the U.V - gauge, 't Hooft and discrete

Successful mirror model must start with sixteen L and sixteen R Weyl fermions.
Must (obviously) have no gauge anomalies
Must achieve gapping using gauge/four fermion interactions

CMT studies have furnished some examples in low dimension using specific quartic interactions

New lattice mirror constructions

- CMT constructions use the Fidkowski-Kitaev interaction $(\chi^T \Gamma_A \chi)^2$
- χ single component fermions in real eight dim spinor rep of $\mathrm{Spin}(7)$
- This interaction can gap out eight Majorana modes in $D = 1$ - number needed to cancel discrete anomaly there.

Staggered fermions are single component

8 copies of REAL staggered field in $D = 4 \rightarrow 16$ L and 16 R Weyl
Identify L with $\epsilon(x) = -1$ and R with $\epsilon(x) = +1$ as $a \rightarrow 0$
Notice similarity spin-Z_4 with lattice symmetry: $\chi \rightarrow i\epsilon(x)\chi$

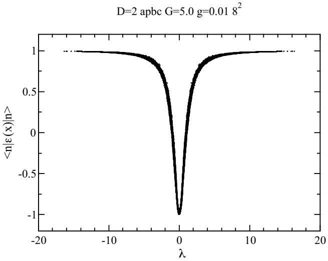
Gap even parity fields yields a continuum limit with just 16 free L Weyl.

Assumes Lorentz invariance is restored.
No (obvious) obstruction to gauging $\mathrm{Spin}(7)$ ...sign problem ?

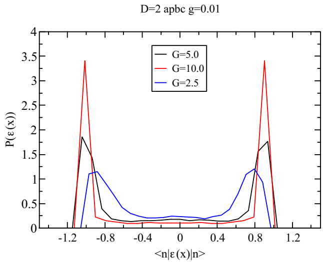
Gapping the even states in $D = 2$

$$\text{Fermion op } M = \eta.D + GP_+\sigma_A\Gamma_A$$

$$< n|\epsilon(x)|n > = < \sum_x \phi_n(x)\epsilon(x)\phi_n(x) > \quad \phi_n \text{ eigenvector of M}$$



(a) $< \epsilon(x) >$ vs eigenvalue



(b) Histogram of parity of modes

Most modes have parity $\epsilon(x) \sim \pm 1$ for large G.
$\epsilon = +1$ modes have large eigenvalue. $\epsilon = -1$ modes are light.

Summary

- Ongoing, exciting and potentially fruitful program of studies in lattice SUSY and chiral fermions.
- Benefited from both theoretical developments and optimized code development.
- Understanding some of the remaining puzzles in SM may require new non-perturbative insights from lattice studies of eg. strongly coupled chiral gauge theories.
- Lattice SUSY + holography may shed light on (quantum) gravity.
- Plenty of room and need for more people to join in !

Thanks!

Backups

Lattice Model – D dims

$$S = S_{\text{kin}} + S_{\text{Yuk}}$$

$$S_{\text{kin}} = \sum_x \sum_{\mu=1}^{D} \sum_{a=1}^{8} \eta_\mu(x)\chi^a(x) D_\mu \chi^a(x)$$

$$S_{\text{Yuk}} = \sum_x \sum_{A=1}^{7} (g_+ P_+ + g_- P_-)\sigma_A(x)\chi^a(x)\Gamma_A^{ab}\chi^b(x) + \frac{1}{2}\sigma_A^2$$

where

$$P_\pm = \frac{1}{2}(1 \pm \epsilon(x)) \text{ with } \epsilon(x) = (-1)^{\sum_{i=1}^{D} x_i} \text{ and } \eta_\mu(x) = (-1)^{\sum_{i=1}^{\mu-1} x_i}$$

$\sigma_A(x)$ real scalars and $\chi^a(x)$ real single component Grassman
Γ_A real, 8×8 antisymmetric Dirac matrices for $\text{Spin}(7)$
$D_\mu f(x) = \frac{1}{2}\left(f(x+\mu) - f(x-\mu)\right)$

Symmetries

Comment

Integrate $\sigma_A \rightarrow$ Fidkowski-Kitaev interaction $-(\chi^T \Gamma_A \chi)^2$.
Gaps Majorana fermions without breaking symmetries in 1d

Global $\mathrm{Spin}(7)$
Global $Z_2 : \chi \rightarrow -\chi$
Discrete shift: $\chi(x) \rightarrow \xi_\rho(x)\chi(x+\rho)$ with $\xi_\rho(x) = (-1)^{\sum_{i=\rho+1}^{D}}$
Discrete rotations (Eucl. Lorentz)

Why $\mathrm{Spin}(7)$?

Want smallest orthogonal group with real spinor representation
Why real ? Crucial for continuum chiral fermions

Symmetries protect theory from all fermion bilinear terms

Continuum Fermions: 2 dims

Assemble staggered fields in unit square into 2×2 matrix fermion (neglect $\mathrm{Spin}(7)$ indices. Site parity shown explicitly)

$$\Psi = \begin{bmatrix} (\chi_+(x) + i\chi_+(x+1+2)) & (\chi_-(x+1) + i\chi_-(x+2)) \\ (\chi_-(x+1) - i\chi_-(x+2)) & (\chi_+(x) - i\chi_+(x+1+2)) \end{bmatrix}$$

Ψ defined on lattice with twice the lattice spacing. Staggered fields coeffs in expansion on products of 2d Dirac matrices $\{I, \sigma_1, \sigma_2, \sigma_1\sigma_2\}$

In continuum: $\Psi \to L\Psi F^T$

Contains two Majorana spinors - defined on even and odd sites

Restore $\mathrm{Spin}(7)$ indices

Continuum: sixteen massless Majorana fermions.
Equivalent to 8 L and 8 R Weyl.

Continuum fermions: 4 dims

Unit hypercube:

$$\Psi(x) = \sum_n \chi(x+n)\gamma_\mu^n$$

with $\gamma^b = \gamma_1^{b_1} \ldots \gamma_4^{b_4}$ and $n_\mu = 0, 1$
Block structure (chiral basis):

$$\Psi = \begin{pmatrix} E & O' \\ O & E' \end{pmatrix} \quad \mathrm{(E,\ E') \ consist\ of\ } \chi_+ \mathrm{\ while\ (O,\ O')\ contain\ } \chi_-$$

Real staggered fields $\rightarrow \Psi^\dagger = \gamma_2 \Psi^T \gamma_2$

Implies $O' = -\sigma_2 O^* \sigma_2$ and $E' = \sigma_2 E^* \sigma_2$
Generalized charge conjugation condition

Yields 2 pairs of Majoranas:

$$\begin{pmatrix} -\sigma_2 O_L^* \sigma_2 \\ O_L \end{pmatrix} \quad \mathrm{and} \quad \begin{pmatrix} E_R \\ \sigma_2 E_R^* \sigma_2 \end{pmatrix}$$

Restore $\mathrm{Spin}(7)$: sixteen L and sixteen R Weyl

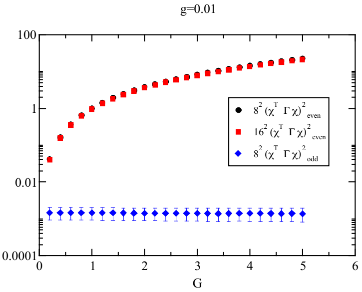
Strategy: Symmetric Mass Generation

- Consider limit $g_+ \to \infty$ and $g_- \to 0$. Will show that even parity fields receive large masses *without* breaking symmetries – Symmetric Mass Generation (SMG). Odd parity fields become massless free fields.
- In continuum ($L \to \infty$) the odd parity fields yield sixteen L Weyl fermions. This number of fermions is no accident !

In 2 dims eight Weyl fermions are needed to cancel off a discrete anomaly - chiral fermion parity $\psi_L \to -\psi_L$ and $\psi_R \to \psi_R$
In 4 dims sixteen Weyls are needed to cancel off spin-Z_4 anomaly $\psi_L \to -i\psi_L$ and $\psi_R \to i\psi_R$

SMG and anomaly cancellation are entwined:
Cannot hope to decouple the mirrors if remaining massless fermions have a non-zero anomaly.

Numerical results: Two dims



Rescale $\chi_- \rightarrow \frac{1}{g_-}\chi_-$, $\chi_+ \rightarrow \frac{1}{g_+}\chi_+$.
Set $\mu = \frac{g_-}{g_+} \rightarrow 0$

$$\eta.D\,\chi_- + \mu(\chi_+\Gamma_A\chi_+)\Gamma_A\chi_+ = 0$$

$$\eta.D\,\chi_+ + \frac{1}{\mu}(\chi_-\Gamma_A\chi_-)\Gamma_A\chi_- = 0$$

Gaps subsets of fermions:

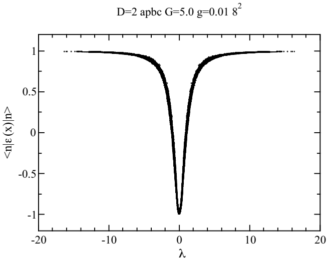
Odd parity sector weakly coupled
Even sector strongly coupled

(with Goksu Can Toga and Nouman Butt)

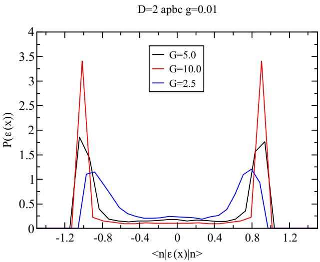
Gapping the even states

$$M = \eta.D + (g_+ P_+ + g_- P_-)\sigma_A \Gamma_A$$

$$< n|\epsilon(x)|n > = < \sum_x \phi_n(x)\epsilon(x)\phi_n(x) > \quad \phi_n \text{ eigenvector of M}$$



(a) $< \epsilon(x) >$ vs λ

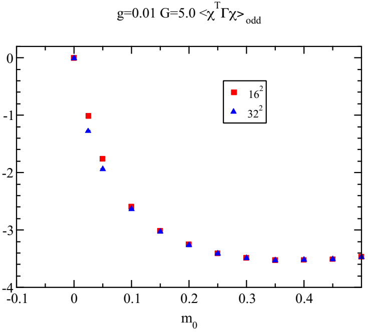


(b) Histogram of parity of modes

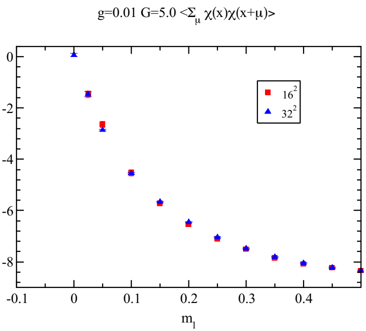
Most modes have parity $\epsilon(x) \sim \pm 1$ for large g_+.
$\epsilon = +1$ modes have large eigenvalue. $\epsilon = -1$ modes are light.

No SSB

Add explicit source terms to action with coupling m and examine limit $m \to 0$ as L increases.



(a) $<\chi^T(x)\Gamma\chi(x)>$ vs source



(b) $<\sum_\mu \xi_\mu(x)\epsilon(x)\chi(x)\chi(x+\mu)>$

No sign of symmetry breaking via bilinear condensates on or off site

Key Points

- Strong lattice four fermi interactions appear capable of separating the eigenstates of the lattice parity operator in eigenvalue space. [Compare with DWF setup which separates eigenstates of γ_5 along 5th dim.]
- In general $\epsilon = -1$ modes are not eigenstates of γ_5. However, provided we impose a reality condition on RSF ϵ becomes a proxy for γ_5 at non-zero lattice spacing and in continuum limit spinors built from χ_- have $\gamma_5 = -1$.
- Absence of spontaneous symmetry breaking consistent with this picture.
- Not clear we need to find new strongly coupled continuous transition. Just drive $g_+ \to \infty$ as $L \to \infty$. Important sector is free and has chiral continuum limit at $g_- = 0$

More on four dimensions

If SMG happens left with pair of massless Majorana fermions in continuum:

$$\begin{pmatrix} -\sigma_2 O_L^* \sigma_2 \\ O_L \end{pmatrix}$$

Can write in terms of just pair of Weyl O_L.
Transform in $(\mathbf{8}, \mathbf{2}, \mathbf{1})$ of $\mathrm{Spin}(7) \times SU(2) \times SU(2)$ global symmetry

Trivial to gauge $\mathrm{Spin}(7)$:

$$D_\mu \chi(x) \to \frac{1}{2}\left(U_\mu(x)\chi(x+\mu) - U_\mu^T(x-\mu)\chi(x-\mu)\right) \to$$

where $U\mu(x) = e^{\frac{1}{4}[\Gamma_A,\Gamma_B]\omega_\mu^{AB}(x)}$

Interaction term already locally $\mathrm{Spin}(7)$ invariant.

Need Wilson term for $U_\mu(x)$ too

chiral lattice gauge theory in continuum limit !

Connection to Pati-Salam GUT

Now Higgs $\text{Spin}(7) \to \text{Spin}(6) \equiv SU(4)$

eg. $\sigma_A^2 = 1$

fermion rep. $\mathbf{8} \to \mathbf{4} + \overline{\mathbf{4}}$

Pati Salam GUT:

fermions $(\mathbf{8}, \mathbf{2}, \mathbf{1})_\mathbf{L} \overset{\text{Spin}(\mathbf{7}) \to \text{Spin}(\mathbf{6})}{\to} (\mathbf{4}, \mathbf{2}, \mathbf{1})_\mathbf{L} \oplus (\mathbf{4}, \mathbf{1}, \mathbf{2})_\mathbf{R}$

Notice: charge conjugation flips internal flavor $(2, 1) \to (1, 2)$

unbroken symmetry: $SU(4) \times SU(2) \times SU(2)$

matter reps+symmetries of Pati-Salam !

Pati-Salam $\rightarrow$ SM

Leptons as fourth color

$$SU(4) \rightarrow SU_c(3) \times U(1)$$
$$4 \rightarrow 3 + 1$$
$$\overline{4} \rightarrow \overline{3} + 1$$

triplet colored quarks + 1 lepton

Left-right symmetric weak interaction

$$SU_W(2) \times SU_{W'}(2) \rightarrow SU_W(2)$$
$$(2,1)_L + (1,2)_R \rightarrow 2_L + 1_R$$

doublet of L fermions and singlet R

Summary

- Reduced (Majorana-like) staggered fermions with carefully chosen Yukawa interactions may realize a mirror model and allow for chiral continuum limit.
- Requires symmetric mass generation (SMG) - non-perturbative physics. Shown evidence in $D = 2$.
- A necessary condition for SMG is cancellation of discrete anomalies - continuum limit of lattice model consistent with this ..
- $\mathrm{Spin}(7)$ symmetry that arises in the model can be gauged - $\rightarrow$ chiral lattice gauge theory
- Connection to Pati-Salam after breaking $\mathrm{Spin}(7) \rightarrow \mathrm{Spin}(6)$.
- SMG remains to be checked by simulation in $D = 4$. Is spectrum really chiral ..? How to tune couplings with L ? Can we decouple chiral modes after switching on gauge interactions ?
- Sign problems likely inevitable - application for QC ?

Thanks!

KD fermions

Generalization of staggered fermions

- Staggered fermions best understood as a discretization of Kähler-Dirac (KD) fermions
- KD equation alternative to Dirac equation. In locally flat backgrounds describes $2^{D/2}$ degenerate Dirac spinors.

Kähler-Dirac equation

$$K\Omega = \left(d - d^{\dagger}\right)\Omega = 0$$

Note: $K^2 = -\Box$. Thus K alternative to $\gamma.D$.

Ω - collection of forms.

More on Kähler-Dirac

From Kähler-Dirac field $\Omega = (\omega_0, \omega_1, \ldots, \omega_D)$ form matrix

$$\Psi = \sum_{p=0}^{D} \omega_{n_1 \ldots n_p(x)} \gamma_1^{n_1} \gamma_2^{n_2} \cdots \gamma_p^{n_p}$$

Can show that the Kähler-Dirac equation:

$$(d - d^\dagger)\Omega = 0$$

equivalent to:

$$\gamma_\mu \partial_\mu \Psi = 0$$

In $D = 4$:

Four copies of Dirac equation where Dirac spinors correspond to columns of Ψ.

Reduced Kähler-Dirac field $\equiv$ real forms.
Implies $\Psi^\dagger = \gamma_2 \Psi^T \gamma_2$

Staggered fermions off the torus ...

Staggered fermions $\rightarrow$ discrete Kähler-Dirac fermions

16 staggered fields in unit hypercube $\rightarrow$ set of antisymmetric tensors $\Omega = (\omega_0, \omega_\mu, \omega_{\mu\nu}, \ldots \omega_{1234})$ associated with *p*-simplices in lattice

Staggered Dirac op $\rightarrow$ discrete Kähler-Dirac operator

$$\eta_\mu D_\mu \rightarrow K = (\delta - \overline{\delta})$$

$\delta, \overline{\delta}$ (co-)boundary operators – analogs of d and $d^\dagger$ eg.

eg. $\delta\,(p - simplex) = \sum(\pm)\ boundary\ (p-1) - simplices$

Discrete Kähler-Dirac equation

$$\left(\delta - \overline{\delta}\right)\Omega = 0 \equiv \textit{staggered equation on regular torus}$$

Valid for any (oriented) random triangulation of any topology. No fermion doubling !