How fast do heavy quarks thermalize in the QGP?

Lattice QCD results for the heavy quark diffusion coefficient

1. **Precise calculation in quenched QCD at $1.5\,T_c$** 10.1103/PhysRevD.103.014511 (2021)

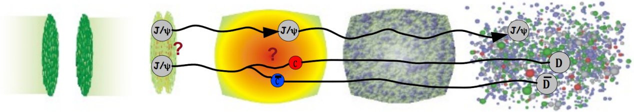
   Bielefeld U.: Altenkort, Kaczmarek, Mazur, Shu
   TU Darmstadt: Eller, Moore

2. **First impressions from $2+1$ flavor QCD**

   Bielefeld U.: Altenkort, Kaczmarek, Shu
   Brookhaven NL: Petreczky, Mukherjee
   U. of Stavanger: Larsen

Luis Altenkort (Bielefeld University)
for the HotQCD collaboration

April 21, 2022
USQCD All Hands' Meeting 2022



What does heavy quark diffusion tell us about the QGP?

- Hydrodynamics ⇒ kinetic equilibration time $\tau_{\text{kin}}^{\text{heavy}} \simeq \frac{M}{T}\,\tau_{\text{kin}}^{\text{light}}$ where $\tau_{\text{kin}}^{\text{light}} \approx \frac{1}{T}$
- But: significant collective motion (v_2)! ⇒ $\tau_{\text{kin}}^{\text{heavy}} \overset{?}{\approx} \frac{1}{T}$ ⇒ $\tau_{\text{kin}}^{\text{light}} \overset{?}{\ll} \frac{1}{T}$
- Knowledge of $\tau_{\text{kin}}^{\text{heavy}}$ essential to understand collisional energy loss and explain exp. data
- Crucial input for quarkonium production models

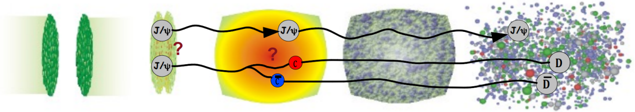
Can we calculate $\tau_{\text{kin}}^{\text{heavy}}$ from first principles?

- Consider non-relativistic limit $M \gg T$:
  (Langevin dynamics)

$$(\tau_{\text{kin}}^{\text{heavy}})^{-1} = \frac{\kappa}{2MT}$$

$$D = 2T^2/\kappa$$

- **Problem:**
  perturbative series for D or κ ill-behaved!
  ⇒ **nonperturbative first-principles** approach: **lattice QCD**



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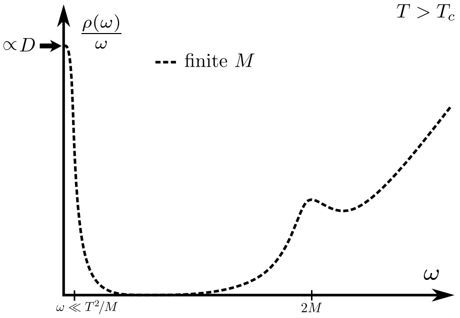
How to calculate diffusion coefficients from the lattice?

- Linear response theory: diffusion physics ⇔ low-energy **in-equilibrium spectral functions** (SPF)
- SPF of HQ vector current $$\boldsymbol{\rho}^{ii}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{i\omega t} \int \mathrm{d}^3\mathbf{x} \left\langle \frac{1}{2}\left[\hat{\mathcal{J}}^i(\mathbf{x},t), \hat{\mathcal{J}}^i(\mathbf{0},0)\right]\right\rangle$$
- reconstruct from **Euclidean correlation functions**: $$G(\tau) = \int_0^{\infty} \mathrm{d}\omega\, \boldsymbol{\rho}(\omega) \frac{\cosh\left(\omega(\tau - \frac{\beta}{2})\right)}{\sinh\left(\omega\frac{\beta}{2}\right)}$$



- fluct.-dissipation: consider Kubo-formula for **momentum diffusion coeff.** $\boldsymbol{\kappa}$ instead of D
- utilize **HQET**: HQ mass $M \to \infty$, expansion in $1/M$, replace $\hat{\mathcal{J}}^i$ with LO versions

⇒ color-electric two-point function (force-force correlator) $G(\tau)$ with

$$\boldsymbol{\kappa} = \lim_{\omega\to 0} 2T \frac{\boldsymbol{\rho}(\omega)}{\omega}$$

⇒ smooth $\omega \to 0$ limit expected: **no transport peak** (much easier to reconstruct)

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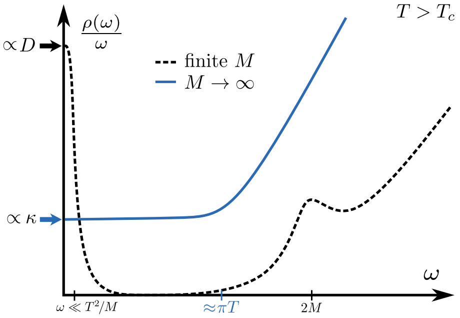

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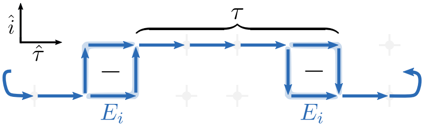
Gluonic color-electric correlator

Caron-Huot et al. 2009

$$\boxed{G(\tau) \equiv \frac{1}{3}\sum_{i=1}^{3} \frac{-\langle \mathrm{Re}\,\mathrm{tr}\;\; U(\beta,\tau)\;\; gE_i(\tau)\;\; U(\tau,0)\;\; gE_i(0)\rangle}{\langle \mathrm{Re}\,\mathrm{tr}\;\; U(\beta,0)\rangle}}$$

$$= \int_0^\infty \mathrm{d}\omega\; \boldsymbol{\rho}(\omega)\, \underbrace{\frac{\cosh\left(\omega(\tau-\frac{\beta}{2})\right)}{\sinh\left(\omega\frac{\beta}{2}\right)}}_{K(\omega,\tau)}, \qquad \boldsymbol{\kappa} = \lim_{\omega\to 0} 2T\frac{\boldsymbol{\rho}(\omega)}{\omega}$$

- Leading order, small τ: $\quad G(\tau) \propto \tau^{-4}$
- Lattice discretization:



The drawback of the $M \to \infty$ limit

- $G(\tau)$ is purely gluonic
  - ⇒ UV gauge fluctuations dominate for large τ
- $K(\omega,\tau)$: large τ are most sensitive to $\omega \to 0$
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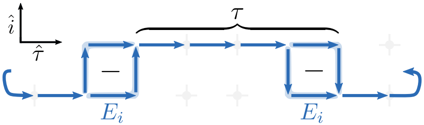

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Solution to gauge noise problem: gradient flow Lüscher 2010

- applicable to nonlocal actions (e.g. 2+1 flavor QCD)
- introduces extra dimension: “flow time” τ_F
- evolves gauge fields $A_\mu(x)$ towards minimum of action S_G

$$A_\mu(x,\boldsymbol{\tau_\mathrm{F}=0}) = A_\mu(x)$$

$$\frac{\mathrm{d}A_\mu(x,\tau_\mathrm{F})}{\mathrm{d}\tau_\mathrm{F}} \sim \frac{-\delta S_G[A_\mu]}{\delta A_\mu(x,\tau_\mathrm{F})}$$

Flow = smooth regulator

- Suppression of high-momentum modes in gluon prop.
- A_μ^LO: average over Gaussian, $\boxed{\text{width} \simeq \sqrt{8\tau_\mathrm{F}} \text{ “flow radius”}}$

$$A_\mu^\mathrm{LO}(x,\tau_\mathrm{F}) = \int \mathrm{d}y \left(\sqrt{2\pi}\sqrt{8\tau_\mathrm{F}}/2\right)^{-4} \exp\left(\frac{-(x-y)^2}{\sqrt{8\tau_\mathrm{F}}^2/2}\right) A_\mu(y)$$

On the lattice

- links replaced by well-defined local averages ⇒ noise suppression
- suppression of renormalization artifacts
- ...but contact terms contaminate $G(\tau)$ for $\boxed{\sqrt{8\tau_\mathrm{F}} \gtrsim \tau/3}$ (LO pert. theory Eller, Moore 2018)

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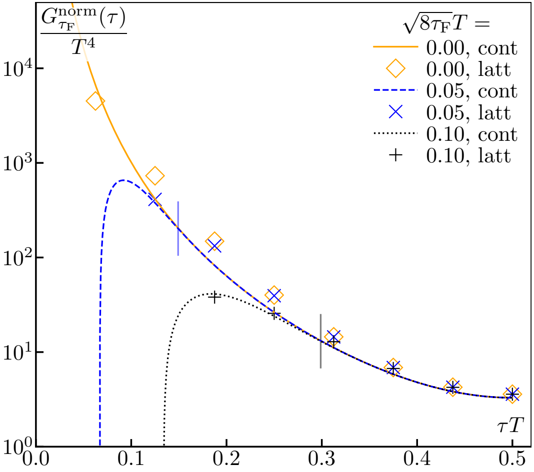
LO perturbative EE correlator (Wilson flow)



continuum corr. from Eller, Moore 2018,
lattice corr. from Eller, Moore, LA et al. 2021

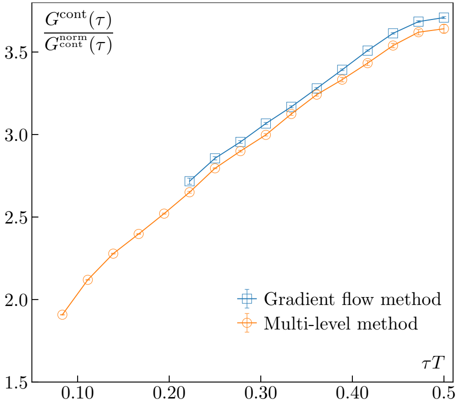
Flow limit = lower bound for τ

- cont. correlator deviates $< 1\%$ for $\boxed{\tau \gtrsim 3\sqrt{8\tau_F}}$ (vertical lines)

Use to enhance nonpert. lattice results:

- get rid of log-scale by normalizing to this
- comparison of LO cont. and LO latt. correlators
  - → remove tree-level discretization errors

Renormalized continuum EE correlator (quenched, $1.5T_c$)



- double-extrapolated EE correlator
- shape consistent with previous pert. renormalized results
  Francis et al. 2015 , Christensen, Laine 2016
- overall shift due to
  - nonperturbative renormalization
  - difference in statistical power of gauge conf.
  - systematic uncertainty introduced by flow extr.
- only large-τ correlator can be obtained

LA et al. 2021

Spectral reconstruction through pert. model fits

- $$G(\tau) = \int_0^\infty \mathrm{d}\omega\, \boldsymbol{\rho}(\omega)\, K(\omega,\tau), \qquad \boldsymbol{\kappa} = \lim_{\omega\to 0} 2T\,\frac{\boldsymbol{\rho}(\omega)}{\omega}$$

⇒ **integral inversion** problem; on paper valid only at $\tau_{\mathrm{F}} = 0$ Eller 2021

- **Strategy:** constrain allowed form of $\boldsymbol{\rho}(\omega)$ using IR and UV **asymptotics**:

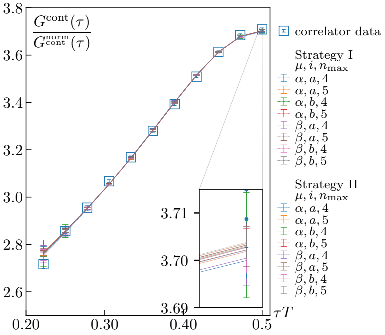
$$\phi_{\mathrm{IR}}(\omega) \equiv \frac{\boldsymbol{\kappa}}{2T}\omega, \quad \phi_{\mathrm{UV}}(\omega) \equiv \frac{g^2(\bar{\mu}_\omega)C_F}{6\pi}\omega^3, \quad \dots$$

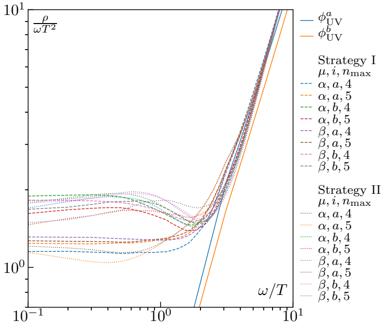
and various **interpolations** $I(w)$:

⇒ $$\rho_{\mathrm{model}}(\omega) \equiv I(\omega)\sqrt{\left[\phi_{\mathrm{IR}}(\omega)\right]^2 + \left[\phi_{\mathrm{UV}}(\omega)\right]^2}$$

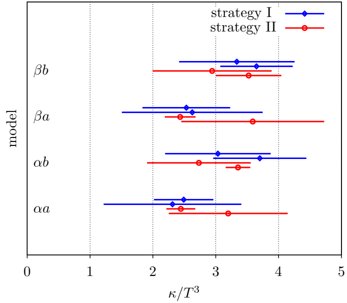
⇒ **well-defined fit** with parameters $\boxed{\boldsymbol{\kappa}/T^3}$ and c_n via

$$\chi^2 \equiv \sum_\tau \left[\frac{G(\tau) - G_{\mathrm{model}}(\tau)}{\delta G(\tau)}\right]^2$$





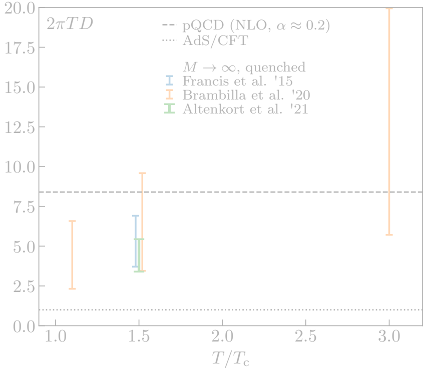
HQ momentum diffusion coefficient (quenched, $1.5T_c$)



- We find $\boxed{\kappa/T^3 = 2.31\ \ldots\ 3.70}$
  - agrees with previous studies
    e.g. Francis et al. 2015 (multi-level method + pert. renorm.)
- convert via $2\pi TD = \dfrac{4\pi}{\kappa/T^3}$: $\boxed{2\pi TD = 3.40\ \ldots\ 5.44}$
- kinetic equilibration time:

$$\boxed{\tau_{\mathrm{kin}} = (1.63\ \ldots\ 2.61)\left(\frac{T_c}{T}\right)^2\left(\frac{M}{1.5\mathrm{GeV}}\right)\ \mathrm{fm/c}}$$

Comparison to previous studies



LA et al. 2021

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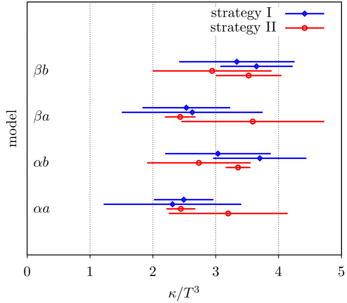

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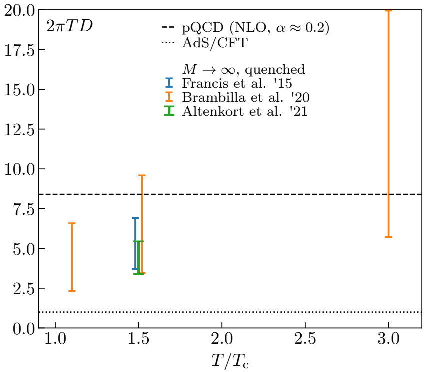


LA et al. 2021

Setup & strategy for $2+1$ flavor QCD

Lattice setup (proposed)

- $2+1$ flavor HISQ fermions
- physical m_s,
  with $m_l = m_s/5$ $(m_\pi \approx 320\,\mathrm{MeV})$

$T\,[\mathrm{MeV}]$	$N_\sigma^3 \times N_\tau$	$a\,[\mathrm{fm}]$
195	$96^3 \times 36$	0.028
	$\mathbf{64^3 \times 24}$	**0.042**
	$\mathbf{64^3 \times 20}$	**0.051**
220	$96^3 \times 32$	0.028
	$\mathbf{64^3 \times 24}$	**0.037**
	$\mathbf{64^3 \times 20}$	**0.045**
251	$96^3 \times 28$	0.028
	$\mathbf{64^3 \times 24}$	**0.033**
	$\mathbf{64^3 \times 20}$	**0.039**
293	$96^3 \times 24$	0.028
	$\mathbf{64^3 \times 22}$	**0.031**
	$\mathbf{64^3 \times 20}$	**0.034**

- + three temp. $\leq 195\,\mathrm{MeV}$
  with **physical masses** $(\mathbf{64^3 \times 24})$

Current situation

- no continuum and flow-time-to-zero extrapolation
- additional resources necessary

However:

- long-distance correlator insensitive to finite (a, τ_F)
- to keep flow corrections small, use τ_F relative to each τ:

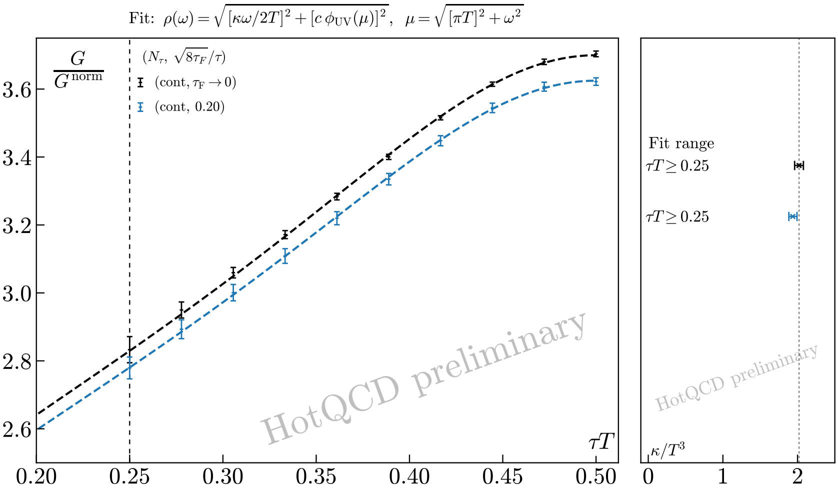
$$\frac{\sqrt{8\tau_F}}{\tau} = \mathrm{const.} < \frac{1}{3}$$

⇛ enough to constrain κ?

So, in the meantime:

- treat finite (a, τ_F) as systematic uncertainties
- use quenched data to estimate additional systematic error

Quenched, $1.5T_c$: systematics of simple model fits



Conclusions

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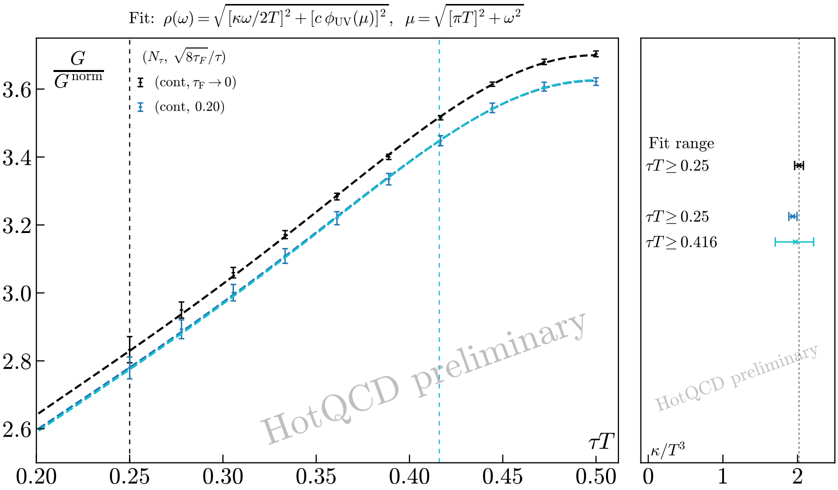

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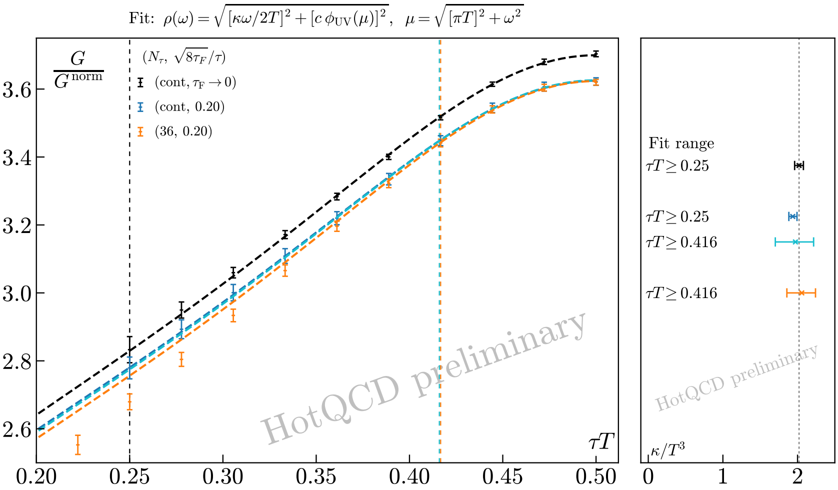

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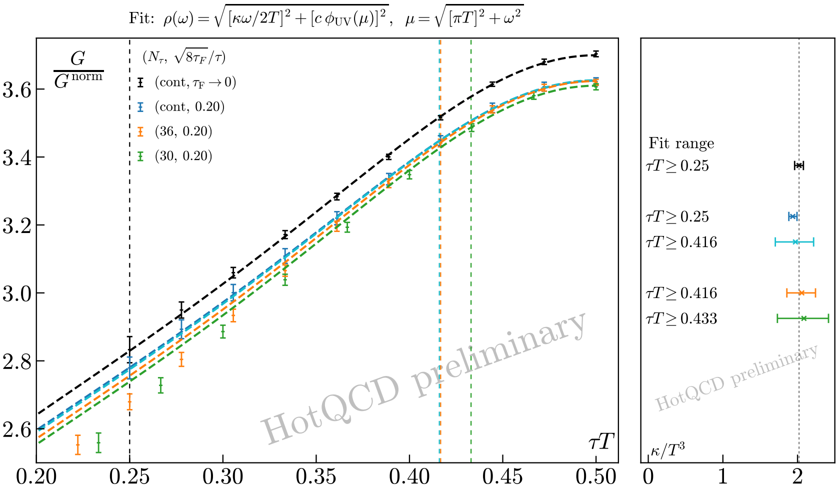

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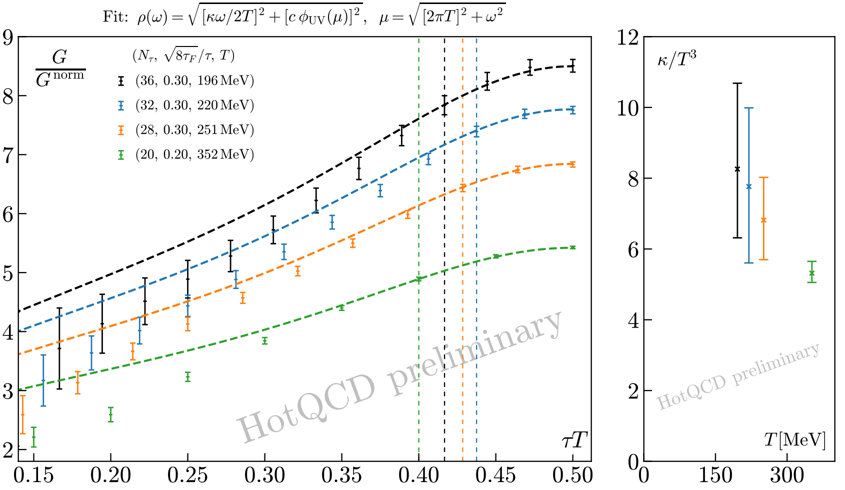
$2+1$ flavor: simple model fits at finite a and τ_F



Lattice setup

T [MeV]	$N_\sigma^3 \times N_\tau$	a [fm]	#
196	$96^3 \times 36$	0.028	1800
220	$96^3 \times 32$	0.028	1700
251	$96^3 \times 28$	0.028	1600
352	$96^3 \times 20$	0.028	4600

- $2+1$ flavor HISQ fermions
- $m_l = m_s/5$ (pion mass $\approx$ 320 MeV)

First impressions

- much larger $\boldsymbol{\kappa}/T^3$ compared to quenched ($\sim 2\times$)
- $\boldsymbol{\kappa}/T^3$ seems to decrease with increasing T

Recap

What do we want?	■ a first-principles nonpert. estimate from full QCD for the **HQ momentum diffusion coefficient** κ (or in turn D, τ_{kin})

Why?	■ phenomenology: explain experimental data for HQ ■ crucial input for transport simulations

What did we achieve so far?

- quenched QCD
  - proof-of-concept for gradient flow method LA et al. 2021
  - consistent results for κ compared to previous studies
  - $(a, \tau_F) \to 0$ data serves as benchmark for systematics of finite (a, τ_F) data
- 2+1 flavor QCD
  - preliminary explorations to constrain κ using finite (a, τ_F) data

What to do next?

- 2+1 flavor QCD
  - increase statistics, add lattice spacings, look into quark mass effects
  - estimate systematic errors
- determine finite-mass correction (color-magnetic correlator) Bouttefeux, Laine 2021