Renormalization of critical ϕ^4 theory on S^2 and $S^2 \times \mathbb{R}$ (USQCD All Hands' Meeting 2022)

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Overview

- Lattice Radial Quantization
- QFE Counterterms
- Modular Ising Model

Lattice Radial Quantization

Lattice Radial Quantization

- We wish to use the lattice to study field theories at or near conformal fixed points
- On a periodic square lattice, wraparound effects are always relevant
- Weyl transform from flat Euclidean manifold to a "cylinder"

$$\mathbb{R}^d \to S^{d-1} \times \mathbb{R}$$

$$ds^2_{\text{flat}} = r^2[(d\log r)^2 + d\Omega^2_{d-1}] \to ds^2_{\text{cyl.}} = dt^2 + d\Omega^2_{d-1}$$

- Angular directions are periodic by definition
- Radial coordinate defined as $t = \log r$
- Power-law correlation functions decay exponentially in t
- Lattice volume grows exponentially with number of time-slices

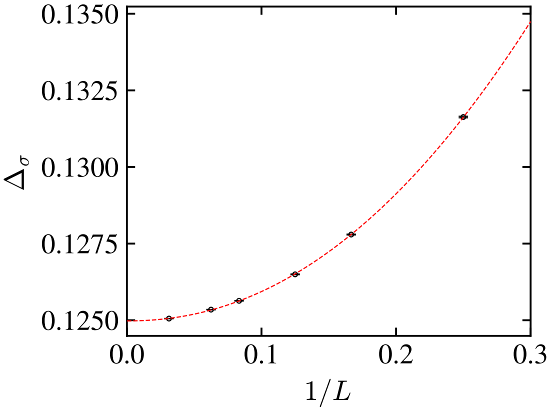
Lattice Radial Quantization

- Discretization in 2d is relatively easy

$$\mathbb{R}^2 \to S^1 \times \mathbb{R}$$

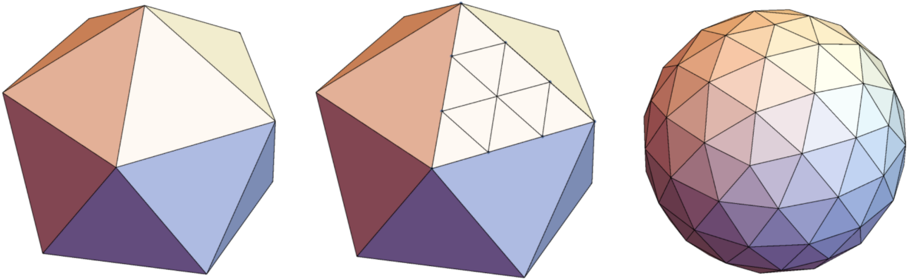
- Scaling exponent for σ operator in 2d Ising model:

$$\Delta_\sigma = 0.1249781(62)$$



Lattice Radial Quantization

- In $d > 2$, angular portion of manifold cannot be discretized uniformly
- In 3 dimensions, $\mathbb{R}^3 \to S^2 \times \mathbb{R}$ can be discretized by tessellating an icosahedron [2]



- Produces a non-uniform simplicial complex
- Higher dimensions can be discretized in a similar manner

Lattice Radial Quantization

- Test case is scalar field theory on S^2 with discretized action

$$S = \frac{1}{2} \sum_{\langle xy \rangle} \frac{V_{xy}}{\ell_{xy}^2} (\phi_x - \phi_y)^2 + \sum_x \sqrt{g_x} \left(\frac{1}{2} m^2 \phi_x^2 + \lambda \phi_x^4 \right)$$

- Free scalar theory ($\lambda = 0$) on a simplicial complex can be solved exactly with the finite element method (FEM)
- Geometric factors V_{xy}, ℓ_{xy}, $\sqrt{g_x}$ are determined by lattice geometry

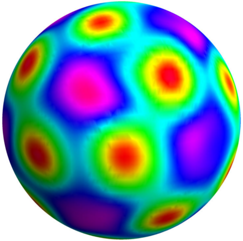
QFE Counter Terms

QFE Counter Terms

- Interacting theory ($\lambda \neq 0$) has UV divergences due to quantum fluctuations from loops
- The quantum theory does not become spherical as $a \to 0$ so conformal symmetry is lost
- At small λ, a perturbative mass counterterm renormalizes the theory [3]

$$S \to S + \frac{1}{2} \sum_x \sqrt{g_x} \delta m_x^2 \phi_x^2$$

- Renormalized coupling must remain fixed in the continuum limit, therefore $\lambda \to 0$ as $a \to 0$



QFE Counter Terms

- Can be used to accurately determine CFT parameters Δ_σ, $\Delta_{\sigma'}$, Δ_ϵ, $\Delta_{\epsilon'}$, etc. [4]
- Scaling exponents for σ' and ϵ' operators are required as inputs for the conformal bootstrap program [7]
- Simulations and data analysis are ongoing for critical ϕ^4 theory in 3 dimensions
- We are also planning to pursue finite element formulations with gauge theories and fermions, preliminary work has been done [5]

QFE Counter Terms

- To first order, one-loop perturbative counter term for ϕ^4 theory on S^2 has the form

$$\delta m_x^2 = 6Q\lambda \log\left(\sqrt{g_x}\right) \qquad Q = \frac{\sqrt{3}}{8\pi}$$

- Conjecture that conformal symmetry can be restored at strong coupling by adjusting strength of counterterms as a function of λ.

$$\delta m_x^2 \to C(\lambda)\delta m_x^2$$

- We obtain good results [1] with $C(\lambda) = e^{-Q\lambda}$. Counterterms become

$$\delta m_x^2(\lambda) = 6Q\lambda e^{-Q\lambda} \log\left(\sqrt{g_x}\right)$$

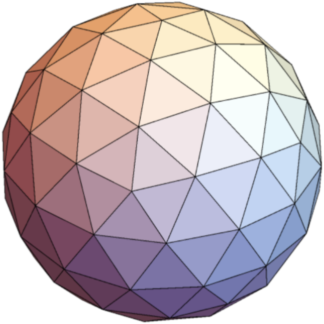
QFE Counter Terms

- Stability of critical surface near the strong coupling Wilson-Fisher fixed point is improved by tuning the strength of the perturbative counterterms.
- Further tuning is required to fully restore symmetry in the continuum limit.
- Exact form of non-perturbative counterterms is not well understood, but may provide direct access to the Wilson-Fisher fixed point.

Modular Ising Model

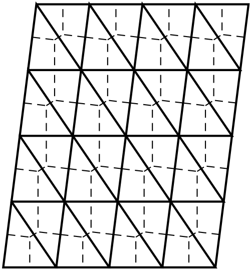
Modular Ising Model

- Approaching the continuum limit, tangent planes of the discretized sphere S^2 become locally uniform, with smoothly varying triangles
- Point defects at 12 "exceptional" points



Modular Ising Model

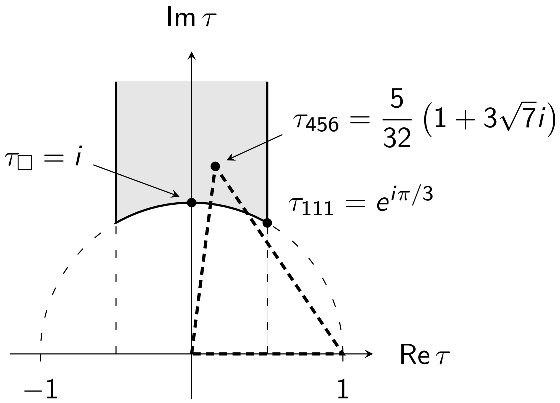
- Critical Ising model can be solved exactly in 2d via free massless fermion [8]
- Wolff [9] relates Ising model to free Majorana fermion on an equilateral triangular lattice with periodic boundaries via a loop expansion
- We generalize this to a uniform lattice of arbitrary triangles to relate Ising couplings (K_1, K_2, K_3) to lattice geometry (ℓ_1, ℓ_2, ℓ_3)



$$S = -\sum_{\langle xy \rangle} K_i \sigma_x \sigma_{x+i} \quad \sinh(2K_i) = \frac{\ell_i^*}{\ell_i}$$

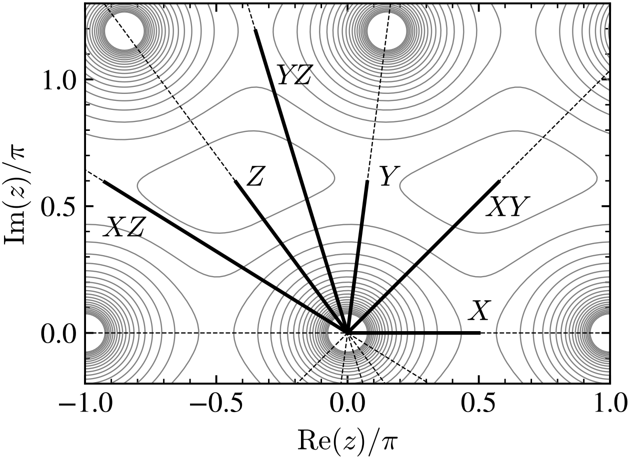
Modular Ising Model

- This allows us to simulate the 2d Ising model on a torus with an arbitrary modular parameter τ (related to ℓ_i's)



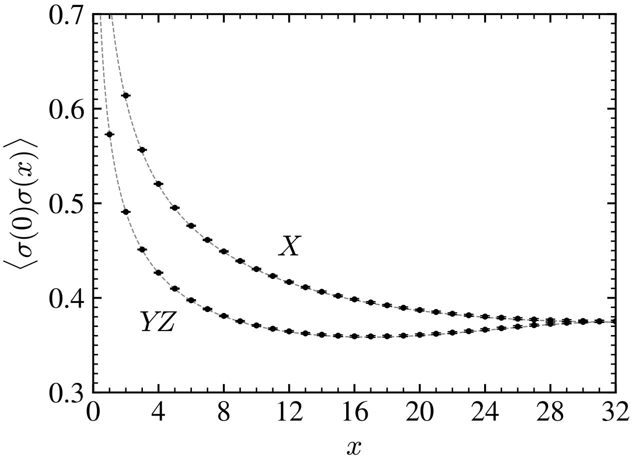
Modular Ising Model

- Continuum spin-spin correlation function is known analytically for arbitrary τ [6], shown for $\ell_i \propto \{4 : 5 : 6\}$



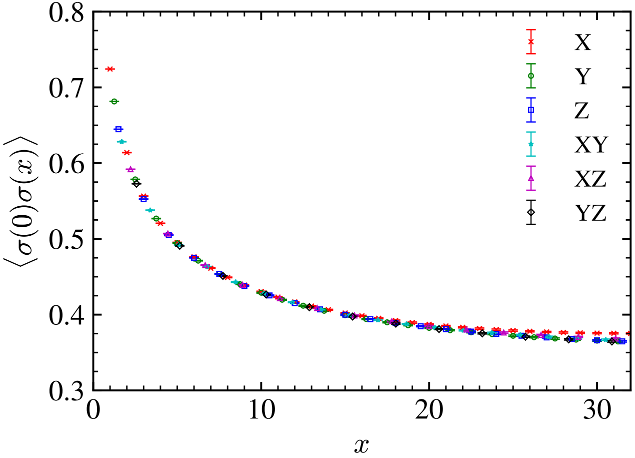
Modular Ising Model

- Our simulation result, with horizontal axis is in lattice steps



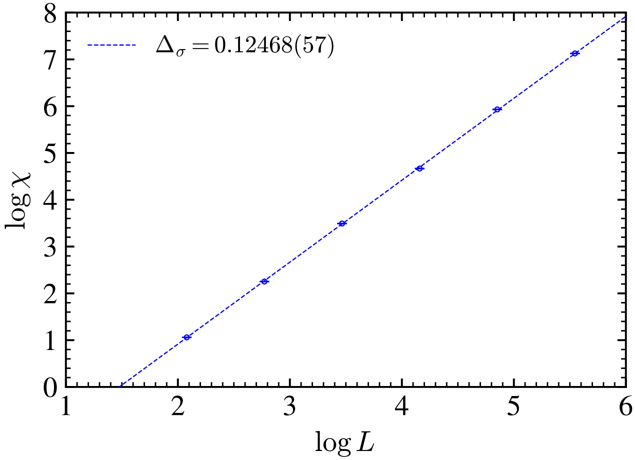
Modular Ising Model

- Horizontal axis scaled according to lattice geometry



Modular Ising Model

- Conventional finite-size scaling analysis on $\{4:5:6\}$ lattice



Modular Ising Model

- We expect that applying critical couplings will give the critical Ising model on S^2, preliminary results support this
- Can also be applied to other 2-dimensional manifolds embedded in $\mathbb{R}^3$

Wrap-Up

- Lattice radial quantization is effective for studying field theories near conformal fixed points
- We are working on several approaches for studying strongly-coupled field theories on curved manifolds using finite element methods
- We have developed a framework for performing simulations of the 2d critical Ising model on a torus with arbitrary modular parameter with potential application to simulations on arbitrary curved manifolds

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